

# INTRODUCTION TO MATHEMATICS

C. C. T. BAKER

MATHEMATICS

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# INTRODUCTION TO MATHEMATICS

By C. C. T. BAKER  
B.Sc., Dip.Ed.

A number system has always been a basic necessity for science, technology and civilization, but the build-up of even the most cumbrous system was a gradual one from the days of the ancient Egyptians, Babylonians and Romans, to the Hindu-Arabic system in use by most people today.

At the present time, however, the pace has altered. Rapid not gradual changes are taking place in education as in other ways of life. New approaches, ideas and techniques are appearing in mathematics which cannot be ignored.

This book has been written expressly to help teacher and student alike to meet the new trends and to be awake to new ways and topics which have recently appeared in some of our schools and colleges.

*Introduction to*  
**MATHEMATICS**

by

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To my Wife



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## PREFACE

Long ago, changes took place slowly and gradually. There was little difference in the way of life, even from one century to another.

At the present time, changes are taking place with increasing rapidity, and there is often a noticeable difference in some aspects of life from year to year.

This applies equally to education. Schools and the subjects taught in them have changed greatly during the lifetime of living people. Today, the whole system of education in this country and others is being altered.

New approaches, new ideas and new techniques are appearing in mathematics. Teacher and student alike should be awake to these new ways so that they can be up-to-date in their knowledge. Many of these changes have come about through change in outlook due to the modern demands of industry and science.

This book deals with some of the topics which have recently appeared in some schools and colleges.

It is hoped that it will be a useful introduction to the new trends.

C.C.T.B.



## CHAPTER ONE

### NUMERATION SYSTEMS

The development of a number system is, and always has been, of fundamental necessity for the purpose of science, technology, and civilization. Even primitive people needed a method of counting. Their method was, what is now called, one-to-one correspondence. Suppose goods exchanged hands, and imagine the goods to be cows. Each cow would be represented by one notch on a stick. Mathematicians now call this matching operation a 'mapping'. In this particular case, each member of the 'set' of cows is mapped on to each member of the 'set' of notches.

This operation also gives us the idea of a 'cardinal' number, because the symbol 8, in 8 cows, is one of the set of cardinal numbers. The 8 is distinct from the 8 in 'the 8th cow', because the latter is what is called an 'ordinal' number.

It took very many centuries to build up even the most cumbersome number system. There have been, in the history of man, many different kinds of number systems. To mention a few, there were those of the Egyptians, Chinese, Babylonians, and Romans. In some systems, each number is represented by an entirely different symbol, each symbol being called a numeral. This makes for a very large number of symbols. The Roman system was, in many ways, very primitive, because the numbers 1, 2, 3, 4, were represented by strokes, |, ||, |||, ||||. The Chinese used horizontal strokes, —, =, ≡, ≡. Today, we use the Hindu-Arabic system. This was invented in India, and it was brought to Europe by the Arabs. Nowadays, most people accept it as being the only possible system of numeration. In fact, it may not be the best, and the time may come when a different and better system will be found. Even now, for some purposes, other systems are already being used. For example, the binary system is employed for computers.

#### PLACE VALUE

The invention of a place value system was the most revolutionary of all, as it reduced considerably the number of symbols needed. In the denary system, which is the one now in common use, the symbols are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. This is



called the base ten system. The zero, 0, was introduced by the Hindus of northern India. Primitive man had no need of a symbol for zero, because he used numerals as recording symbols, not as working symbols. Zero is essential in any place value system.

Now to explain the meaning of 'place value'. Consider the denary number 333. Each 3 has a different value. The 3 on the left stands for 300, the 3 in the middle stands for 30, and the 3 on the right stands for 3 units. In fact, the number 333 could be written as:

$$333 = 3 \times 10^2 + 3 \times 10^1 + 3 \times 10^0.$$

Similarly, a number such as 5387 could be written as:

$$5387 = 5 \times 10^3 + 3 \times 10^2 + 8 \times 10^1 + 7 \times 10^0.$$

It will be seen that, as a number moves towards the left its value increases tenfold, and the indices belonging to the base 10 increase by one.

The denary system is sometimes called the decimal system, and decimal fractions can be treated like the integers, so the fraction 0.8349 could be written as:

$$0.8349 = 8 \times 10^{-1} + 3 \times 10^{-2} + 4 \times 10^{-3} + 9 \times 10^{-4}.$$

It could be mentioned here, that the 0, in 0.8349, may be regarded as a place holder. Its use will be seen to be very necessary in a number such as 8309. This Hindu-Arabic numeration system has vast superiority over the Roman system, especially in, even simple, multiplication and division.

#### THE DUODECIMAL SYSTEM

This system is based on the number 12, mainly because there are 12 inches in one foot. It is often used by carpenters and by builders. There are, however, other uses for a base 12, as in (a) 12 × 12 make a gross, (b) 12 eggs make one dozen, (c) 30 × 12 degrees make one revolution, (d) 5 × 12 minutes make one hour.

If a duodecimal system were universally adopted, it would be necessary to introduce two new symbols, one for ten, say *t*, and one for eleven, say *e*. The twelve symbols would then be:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *t*, *e*.

## CHAPTER TWO

### THE BINARY SYSTEM

This system was first advocated in the seventeenth century, by the German mathematician Leibnitz, because it uses only two symbols, 0 and 1. Strangely enough, some very primitive people used it long before the seventeenth century.

Like so many topics in mathematics, binary numbers dropped out of interest for a long time, but the appearance of the electronic computer has brought new uses for them. This is the reason. In an electronic circuit, there are two states to consider. Either a current is flowing, or it is not flowing. Thus, the number 1 can represent a circuit with the switch 'on', and the number 0 can represent a circuit with the switch 'off'.

The principle of place value is used with binary numbers in the same manner as it is used with denary numbers. Using the denary system, the place values are such as

$$10^4 + 10^3 + 10^2 + 10^1 + 10^0,$$

etc. Using the binary system, the place values are such as  $2^4 + 2^3 + 2^2 + 2^1 + 2^0$ , etc.

TABLE CONNECTING DENARY AND BINARY NUMBERS

<i>Denary form</i>	<i>Index form</i>	<i>Binary form</i>
1	$1 \times 2^0$	1
2	$1 \times 2^1 + 0 \times 2^0$	10
3	$1 \times 2^1 + 1 \times 2^0$	11
4	$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	100
5	$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	101
6	$1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$	110
7	$1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$	111
8	$1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	1000
9	$1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	1001

When a number is written down, only the coefficients are stated, as in the comparison table on page 4.



PLACE VALUE COMPARISON TABLE

Denary			Binary							
100	10	1		64	32	16	8	4	2	1
		1							1	1
		2							1	0
		3							1	1
	1	5					1	1	1	1
	2	9				1	1	1	0	1
	3	7			1	0	0	1	0	1
	5	1			1	1	0	0	1	1
1	2	3		1	1	1	1	0	1	1

## TO CHANGE FROM DENARY TO BINARY NOTATION

*Method.* Find the largest multiple of 2 in the number, subtract this multiple of 2, repeating the process with the remainder.

*Example.* Change 13 in the denary system to the binary system.

- The highest multiple of 2 in 13 is 8, which is  $1 \times 2^3$ . Subtracting this multiple of 2:  $13 - 8 = 5$ .
- The highest multiple of 2 in 5 is 4, which is  $1 \times 2^2$ . Subtracting this multiple of 2:  $5 - 4 = 1$ .
- Therefore,  $13_{10} = 1101_2$ .  $13_{10}$  means 13 in the denary system. So  $13_{10}$  is the same as  $1101_2$ , which represents 1101 in the binary system.

## Further Examples:

- $5_{10} = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 101_2$ .
- $17_{10} = 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 10001_2$ .
- $38_{10} = 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 100110_2$ .

## EXERCISE 1

Change the following denary numbers to binary numbers:

- 7
- 9
- 12
- 20
- 38
- 50
- 53
- 64
- 121
- 256

## ANSWERS TO EXERCISE 1

- 111
- 1001
- 1100
- 10100
- 100110
- 110010
- 110101
- 1000000
- 1111001
- 100000000

## TO CHANGE FROM BINARY TO DENARY SCALE

- Method.* (i) Multiply the left-hand digit by 2.  
(ii) Add on the next digit to the right.  
(iii) Multiply the result by 2.  
(iv) Add on the next digit to the right.  
And repeat this process.

*Example.* Change the binary number 111 to a denary number.

Write the number down. 1 1 1.

- Multiply the left-hand digit by 2 and get 2.
  - Add the 1 to the right:  $2 + 1 = 3$ .
  - Multiply the result by 2:  $3 \times 2 = 6$ .
  - Add on the next 1 to the right:  $6 + 1 = 7$ .
- Therefore,  $111_2 = 7_{10}$ .

## Further Examples:

- $11_2 = 1 \times 2 + 1 = 3_{10} \therefore 11_2 = 3_{10}$
- $1101_2 = 1 \times 2 + 1 = 3$   
 $3 \times 2 + 0 = 6$   
 $6 \times 2 + 1 = 13 \therefore 1101_2 = 13_{10}$

## EXERCISE 2

Change the following binary numbers to denary numbers.

- 101
- 110
- 1101
- 10111
- 11011
- 11111
- 110111
- 111101
- 10·1
- 11·01

## ANSWERS TO EXERCISE 2

- 5
- 6
- 13
- 23
- 27
- 31
- 55
- 61
- $2\frac{1}{2}$
- $3\frac{1}{4}$

## OPERATIONS INVOLVING BINARY SCALE NUMBERS

## (1) ADDITION

The main points to remember are:

- $1 + 0 = 1$
- $1 + 1 = 10$ .

*Example 1.* 11  
10

101

$$\therefore 11 + 10 = 101$$

*Example 2.* 1011  
101

10000

$$\therefore 1011 + 101 = 10000$$



### EXERCISE 3

Add the following binary numbers.

1.  $11 + 11$       2.  $100 + 11$       3.  $110 + 11$       4.  $111 + 11$
5.  $1001 + 1$       6.  $1101 + 10$       7.  $1011 + 10$       8.  $110110 + 101$
9.  $1111 + 111$       10.  $11101 + 11$ .

### ANSWERS TO EXERCISE 3

1. 110      2. 111      3. 1001      4. 1010      5. 1010
6. 1111      7. 1101      8. 111011      9. 10110      10. 100000.

### (2) SUBTRACTION

The main points to remember are:

- (a)  $1 - 1 = 0$     (b)  $1 - 0 = 1$     (c)  $10 - 1 = 1$ .

Example 1.  $101$

$$\begin{array}{r} 101 \\ 11 \\ \hline 10 \end{array}$$

Example 2.  $110$

$$\begin{array}{r} 110 \\ 11 \\ \hline 11 \end{array}$$

Example 3.  $11100$

$$\begin{array}{r} 11100 \\ 101 \\ \hline 10111 \end{array}$$

### EXERCISE 4

Subtract the following binary numbers.

1.  $111 - 100$       2.  $111 - 11$       3.  $110 - 101$
4.  $100 - 11$       5.  $1101 - 111$       6.  $1010 - 101$
7.  $1000 - 111$       8.  $10001 - 111$       9.  $1000 - 101$
10.  $10000 - 1001$ .

### ANSWERS TO EXERCISE 4

1. 11      2. 100      3. 1      4. 1      5. 110
6. 101      7. 1      8. 1010      9. 11      10. 111.

### (3) MULTIPLICATION

The main points to remember are:

- (a)  $1 \times 0 = 0$     (b)  $0 \times 1 = 0$   
 (c)  $1 \times 1 = 1$     (d)  $0 \times 0 = 0$ .

The process is the same as that applied to ordinary denary numbers. Also, to multiply by 10, simply add a 0.

Thus,  $101 \times 10 = 1010$ . Also  $101 \cdot 101 \times 100 = 10110 \cdot 1$ .

Example 1.  $101 \times 11$ .

$$\begin{array}{r} 101 \\ 11 \\ \hline 1010 \\ 101 \\ \hline 1111 \end{array}$$

Example 2.  $111 \times 101$

$$\begin{array}{r} 111 \\ 101 \\ \hline 11100 \\ 111 \\ \hline 100011 \end{array}$$

### EXERCISE 5

Multiply the following binary numbers.

1.  $1011 \times 101$       2.  $1011 \times 110$       3.  $1001 \times 111$
4.  $1011 \times 111$       5.  $1111 \times 1100$       6.  $1110 \times 1101$
7.  $11001 \times 1101$       8.  $11 \times 1 \cdot 1$       9.  $1101 \times 1 \cdot 01$
10.  $1010 \times 11 \cdot 01$

### ANSWERS TO EXERCISE 5

1. 110111      2. 1000010      3. 111111      4. 1001101
5. 10110100      6. 1111110      7. 101000101      8. 100 \cdot 1
9. 10000 \cdot 01      10. 100000 \cdot 10.

### (4) DIVISION

The four operations on binary numbers are all simpler, but longer than on denary numbers. Division proceeds the same in both scales. To divide by 10, the decimal point is moved one place to the left, and to divide by 100, the decimal point is moved two places to the left.

Example 1.  $1100 \div 10 = 110$

Example 2.  $1011 \div 1000 = 1 \cdot 011$

Example 3.  $101)11010(101$

$$\begin{array}{r} 101 \\ 110 \\ \hline 101 \\ \hline 1 \end{array}$$

$\therefore 11010 = 101$  and remainder 1.



# EXERCISE 6

Divide the following binary numbers.

1. 10100  $\div$  101
2. 1001  $\div$  100
3. 10101  $\div$  11
4. 10010  $\div$  110
5. 11000  $\div$  110
6. 10101  $\div$  11
7. 1111  $\div$  101
8. 11110  $\div$  101
9. 10110  $\div$  1011
10. 11011  $\div$  1001.

## ANSWERS TO EXERCISE 6

1. 100
2. 10-01
3. 111
4. 11
5. 100
6. 111
7. 11
8. 110
9. 10
10. 11.

# CHAPTER THREE

## CALCULATING DEVICES

Calculating devices have a very long history, dating from the time of primitive man. The first calculating machine was probably the human hand. It is quite possible that the denary system owes its existence to the fact that the human being has ten fingers. Most, if not all, children begin to count with the aid of the fingers. The abacus, an arrangement of wires holding rows of coloured beads, has been in existence for centuries. It is still popular with children. The early Chinese used the abacus, and other kinds of counting boards. More recently, John Napier invented a device called Napier's rods, or Napier's bones. Napier invented one of the first exponential calculators, the natural logarithm, the base being  $e$ . John Briggs developed common logarithms, to base 10. The slide rule, so much used by engineers, is an exponential calculator, and its use depends upon the laws of indices.

(1) Since  $a^m \times a^n = a^{m+n}$ , it will be seen that, if the logarithms of numbers are added, the logarithm of the product is obtained.

(2) Since  $a^m \div a^n = a^{m-n}$ , it will be seen that if the logarithms of numbers are subtracted, the logarithm of the quotient is obtained. The slide rule makes use of this property of logarithms.

## COMPUTERS

As civilization has advanced new apparatus and new devices, new materials and new processes, have been invented. Now, even space outside our planet is being explored. The means for doing these things have depended more and more on complicated and lengthy calculations. Some of the calculations involved would have required thousands of mathematicians working for many years. Fortunately, calculations can now be performed extremely rapidly with a computer, for it is capable of performing highly complex operations at the speed of light. Although a computer is a very complicated piece of equipment, it can do only three things: (i) add two numbers, (ii) subtract one number from another number, and (iii) compare the size of one number with that of another.

Computers are of two types, the analogue computer, and the digital computer.

1. The analogue computer makes use of variations in electric current and voltage to represent numerical values. Very often, in the physical world, quantities are connected by the same kind of formula, even though the quantities bear no connection with each other. Thus, Newton's Law,  $p = mf$ , connecting force, mass, and acceleration, is the same kind of formula as Ohm's Law,  $V = IR$ , connecting voltage, current, and resistance. This means that an electric circuit, by varying current and voltage, can measure variations in an 'analogous' system. Analogue computers, however, are less accurate than digital computers.

2. The digital computer performs calculations as one would do problems in arithmetic, using an abacus. Before a computer can deal with a problem the data has to be 'programmed' by a mathematician known as a programmer. He expresses the mathematical steps involved into a code or language suitable for the machine to deal with. This may be done on punched cards, on paper tape, or on magnetic tape. Although punched cards can be bulky, they are useful for keeping a record of data constantly being used. Paper tape is fragile but it is less bulky, it can be read quickly, but it cannot be altered if a mistake has been made. Magnetic tape usually consists of a ferric oxide fixed between two layers of a plastic. It is about 0.5 in. wide, and contains about 9 channels. The code consists of a series of magnetized spots, corresponding to the holes on a paper tape. The programmer breaks down the problem into a succession of additions and subtractions; multiplication being carried out by repeated addition. Some computers can add, or subtract, hundreds of thousands of times each second. The main parts of a computer are the arithmetic unit in which the actual calculating takes place, a storage or memory unit where previous results are collected, and a control unit which ensures that the calculations are performed in the correct order.

The parts of a computer are extremely complicated, and resemble thousands of television circuits.

Figure 1 shows a very simple arrangement of the units. The problem may be mathematical, commercial, industrial, or even human, such as managerial. Whatever it is, it is first codified by the programmer. These codes are split up into small steps and are fed into the machine in the form of electrical impulses, similar to the dialling of a telephone. The impulses are created

by the input unit: they are generated by the programme which is constructed like a typewriter, and are recorded on the tape which may be of paper or of metal.

The control unit arranges the route which the impulses will take, and it automatically prepares the parts of the machine which will be needed, and cuts out those which will not be required. The memory unit stores information which is being continually used.

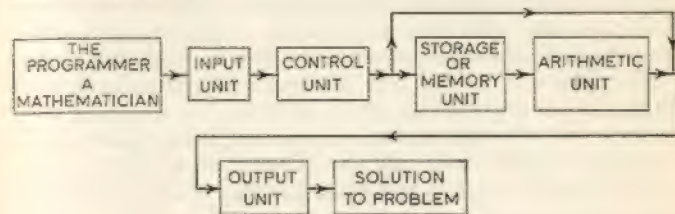


FIG. 1

The solution to the problem eventually reaches the output unit in the form of electrical impulses which are changed into holes in paper tape, into magnetized spots on steel tape, or into holes in punched cards. It then needs a programmer to decode the result to give the solution to the problem.



## CHAPTER FOUR

### THE BASIC LAWS OF MATHEMATICS

#### ADDITION

We already possess the set of symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, which are called numbers. If two members of this set are chosen then, associated with this pair, there is another member called the sum. Let the numbers 3 and 5 be chosen, then their sum is 8. The three and the five are chosen in a certain order, and we then have the ordered pair (3, 5). Addition is a mapping, since it maps a set of ordered pairs into a set of single numbers. The mapping characteristic can be illustrated symbolically as  $(3, 5) \xrightarrow{+} 8$ . The sign  $+$  indicates that the process of addition mapping is performed.

A mapping which assigns to each ordered pair of objects in a set another object in the same set is called a binary operation. Since the sum is in the same set as the element, the operation is said to be closed under addition.

Since there are many ordered pairs which can be mapped on to the same image, for example,  $(2, 6) \xrightarrow{+} 8$ , addition is said to be a many-to-one mapping.

#### COMMUTATIVE LAW OF ADDITION

The ordered pairs (3, 5) and (5, 3) have both the same image. If the members of an ordered pair change places, they are said to commute. Since the sum is the same, addition of the natural numbers is a commutative operation. Symbolically, the commutative law of addition is expressed as

$$a + b = b + a.$$

This statement may appear trivial, but that is because we are accustomed to the fact that, say,  $6 + 4 = 4 + 6$ . However, familiarity can be misleading, and it is important to point out that not all the elementary rules of arithmetic are commutative. For example, division is not commutative because  $\frac{8}{4}$  is not the same as  $\frac{4}{8}$ . It is not possible to commute the 8 and the 4.

#### THE ASSOCIATIVE LAW OF ADDITION

Consider the sum of the three numbers 4, 6, 8. It may be found in two ways. (i) Add the 4 and the 6 to make 10, and then

add the 8 to this total to make 18, which is the sum. (ii) Add the 6 and the 8, to make 14, and then add the 4 to this total to make 18, again. This means that it is possible to associate the numbers in any order without affecting their sum. This is called the associative law for addition. Expressed symbolically, it states that:

$$a + b + c = (a + b) + c = a + (b + c).$$

Addition is only one of the binary operations which possesses this associative property, but not all binary operations possess it. For example, the process of finding the arithmetic mean, or average, of numbers does not obey the associative law.

Let the symbol  $\oplus$  be used to stand for 'find the average'.

Then (i)  $6 \oplus 8 = 7$  and (ii)  $4 \oplus 8 = 6$ .

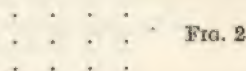
However, (a)  $(6 \oplus 8) \oplus 4 = 7 \oplus 4 = 5\frac{1}{2}$ ,

and, (b)  $6 \oplus (8 \oplus 4) = 6 \oplus 6 = 6$ .

Therefore,  $(6 \oplus 8) \oplus 4$  is not the same as  $6 \oplus (8 \oplus 4)$ .

#### MULTIPLICATION

Young children often perform the operation of multiplication by drawing a rectangular array of dots, the number of rows corresponds to one of the numbers to be multiplied and the number of columns corresponds to the other number. For simplicity, suppose it is required to multiply 3 by 4. Then, the arrangement of dots would be as in Fig. 2.



The total number of dots gives the product of the two numbers. In general, to multiply the number  $a$  by the number  $b$ , it is necessary to find the cardinal number of a set consisting of  $a$  rows each containing  $b$  objects.

A product is represented by a dot, or by the sign  $\times$ , so the product of  $a$  and  $b$  is  $a.b$  or  $a \times b$ .

This process of finding a product is a mapping of natural numbers into a natural number, and it is expressed symbolically as  $(a, b) \xrightarrow{\times} a.b$ . As an example,  $(3, 4) \xrightarrow{\times} 12$ .

Since this mapping is defined for every ordered pair of natural numbers, and the image is always a natural number, the operation of multiplication is a binary operation.



# THE COMMUTATIVE LAW OF MULTIPLICATION

It is well known that numbers may be multiplied in any order, because  $3 \times 4 = 4 \times 3$ , and  $5 \times 6 = 6 \times 5$ , so that, generally,  $a \times b = b \times a$ , or  $a.b = b.a$ . This is the commutative law of multiplication.

# THE ASSOCIATIVE LAW OF MULTIPLICATION

Extending the process of multiplication to three numbers, it is well known that,

$$(i) \quad 3 \times (4 \times 5) = 3 \times 20 = 60 = (3 \times 4) \times 5.$$

Also, (ii)  $4 \times (5 \times 7) = 4 \times 35 = 140 = (4 \times 5) \times 7$ , and so on.

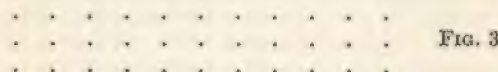
In general  $a \times b \times c = a \times (b \times c) = (a \times b) \times c$ .

This is the associative law of multiplication.

# THE DISTRIBUTIVE LAW OF MULTIPLICATION

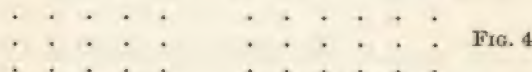
This law forms a link between the multiplication and addition of natural numbers, and expresses the fact that multiplication is distributive with respect to addition. The law can be illustrated by the drawing of a rectangular array of dots.

Consider an array consisting of 3 rows, each row containing 11 dots (Fig. 3).



The total number of dots is  $3 \times 11 = 33$ .

This single array may be split up into two arrays (Fig. 4).



One array has 3 rows, each containing 5 dots and the other has 3 rows each containing 6 dots.

The total number of dots is then  $3 \times 5 + 3 \times 6$ .

Therefore  $3 \times 5 + 3 \times 6 = 3 \times 11$ ,

or,  $3 \times (5 + 6) = 3 \times 5 + 3 \times 6$ .

In general, if  $a$ ,  $b$ , and  $c$  are natural numbers,

$$a \times (b + c) = a \times b + a \times c.$$

This is the distributive law, and it shows that the multiplier can be distributed among the individual terms of the addition.

In this law, however, multiplication and addition cannot change places, because addition is not distributive with respect to multiplication.

This means that:  $3 + (5 \times 6)$  is not the same as

$$3 + 5 \times 3 + 6.$$

The above laws have been explained by using natural numbers as examples. However, the laws would hold for any other abstract set of symbols, because cardinal numbers have been used which are completely detached from objects of any kind. This shows that it is possible to construct a very large variety of number systems, and these may all be defined thus:

A number system is any collection of objects on which binary operations called addition and multiplication are defined such that addition is commutative and associative, multiplication is also commutative and associative, and multiplication is distributive with respect to addition.

# ZERO

This is a very important concept. Early man had no use for zero, but with the invention of place value in numeration systems, it became necessary to have a 'place holder'. Zero was first thought of by the Hindus, and, later, the Arabs represented it by the symbol 0. When zero became a part of our present number system, it meant that zero had to behave in a manner which was consistent with that number system. Zero has the following properties:

1. Zero plus any number gives the same number again.

i.e.,  $0 + x = x$ .

2. Zero times any number gives zero.

i.e.,  $0 \times x = 0$ .

Although zero has the above properties for our number system, they may not be true for other number systems. From property 1, zero may be called the 'identity element' for addition because,

$$x + 0 = 0 + x = x, \text{ for all } x \text{ (written } \forall x).$$

# UNITY

It has been shown that if the zero element is added to another element, the other element remains unchanged. There is another natural number which bears a similar relationship to the process of multiplication. This is the number 1, the unity element. It obeys the rule:

$$1 \times x = x, \text{ for all } x.$$

Thus, multiplying a number by 1 leaves that number unchanged. Elements, in whatever number system, which behave like this are called unity elements.

The identity law for multiplication is,

$$x \cdot 1 = 1 \cdot x = x, \text{ for all } x \text{ (written } \forall x)$$

#### ZERO AND UNITY COMPARED

Both of these have the same property. One has it in relation to one operation, and one to another operation. They are both examples of identity elements, because, if, in any system possessing a binary operation symbolized by  $\oplus$ , there is an element  $e$  which has the property  $e \oplus x = x$ , for all  $x$  in the system, then  $e$  is called an identity element.  $e$  comes from the German word *einheit*, which means unity.

## CHAPTER FIVE

### SETS

There is nothing difficult about the concept of a set. It is simply a collection or a group of things, people or numbers. The term could be applied to a family, or a team, or a pack of cards, a flock, a school, a nation, the consonants, and so on.

Those items which belong to a set are called its members, or its elements. It is not necessary for the things in a set to have any connection with each other, but it must be possible to distinguish members from non-members.

The theory of sets was probably first developed by Cantor, the German mathematician. It did not receive much support at first, but recently its use has been extended to many branches of mathematics, industry, commerce, and logic.

#### SET NOTATION

As the general study of set theory has only recently begun to be studied, there is as yet no standardized symbolism, and different authors often use different notation. However, that adopted here is the most commonly used.

A set is symbolized by a pair of braces or curly brackets  $\{ \}$ , or by parentheses  $( )$ , or square brackets  $[ ]$ . Members of the set are either placed inside or they are described inside. Thus, the set of odd numbers between nought and ten would be indicated as  $\{1, 3, 5, 7, 9\}$ . The set of multiples of 5 from 0 to 31 may be indicated as  $\{5, 10, 15, 20, 25, 30\}$ . A set may be specified by stating some property of its members, such as  $\{e, e \text{ is a circle}\}$ .

Sometimes a single capital letter is used to denote a set.  $A$  could denote all the vowels, so  $A = \{a, e, i, o, u\}$ .  $E$  could mean a set of even numbers,  $E = \{2, 4, 6, 8, 10\}$ . The method of describing a set should show clearly which items belong to the set and which do not.

#### MEMBERSHIP OF A SET

The symbol used for 'is a member of' is  $\in$ .  $\in$  is the Greek  $\epsilon$ , and it is used because  $e$  occurs in 'is an element of', element and member being synonymous.

If the set  $N$  is the set of natural numbers 1 to 9, so that  $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ , then it may be said that  $6 \in N$ . It may also be said that  $7 \in N$ .



The symbol used for 'is not a member of' is  $\notin$ . Therefore, it may be said that  $0 \notin N$ . Similarly, if  $A$  stands for the set  $\{a, e, i, o, u\}$ , then  $b \notin A$ .

#### KINDS OF SET

1. A set is *countable*, or *denumerable*, if all its members can be arranged in some order, one member being first, one second, and so on.
2. A *finite set* has a definite number of members. It is not necessary for the actual number to be known. The number of grains of sand on the earth, although immense in number, is finite. All finite sets are countable, at least in theory.
3. An *infinite set* has an infinite number of members. The set of whole numbers is an infinite set. It does not follow that all infinite sets are uncountable, or non-denumerable. The set of points on a line, or the set of prime numbers, or the set of odd numbers, are examples of infinite sets.
4. The *empty set*, or the *null set*, has no members. It is denoted by  $\{ \}$ , or by the symbol  $\emptyset$ . It is similar to zero in the common numeration system. Men 10 ft tall, cats with four tails, triangles with four sides, are empty sets.
5. The *universal set*, or the *universe*, is denoted by the capital letter  $U$ . This set contains all members, although it has to be described carefully and specifically. The following could be universal sets: all the pupils in a class, all Englishmen, all the books in a library.
6. *Subsets* are sets within sets. A subset is part of a set. The set of pupils with brown eyes would be a subset of the set of pupils in a school. The set of pupils who studied Latin would also be a subset of all the children in a school. The set of pupils with brown eyes could be a subset of those pupils who study Latin. Also, those who study Latin could be a subset of those with brown eyes.

The symbols for a subset are  $\supset$  and  $\subset$ . The symbol  $\supset$  means 'is a subset of', and the symbol  $\subset$  means 'has as one of its subsets'.

Let the set  $A = \{1, 2, 3, 4, 5, 6\}$ , and let the set  $B = \{1, 2, 3, 4\}$ .

Then  $B \supset A$ , and  $A \subset B$ .

If  $T$  is the set of all triangles (an infinite set), and if  $E$  is the set of all equilateral triangles (also an infinite set), then  $E \supset T$ , and  $T \subset E$ .

7. *Equal sets*. Two sets are compared by comparing the

members of one set with the members of the other. The set  $\{a, b, c, d\}$  and the set  $\{p, q, r, s\}$  are different because their members are different. However, the set  $\{a, b, c, d\}$  and the set  $\{c, d, a, b\}$  are equal sets because they have the same members.

It does not matter if the members are not in the same order. Any two sets with the same members are equal. If the members of a set are listed it is easy to compare them. Comparison is not so easy if the members are merely described.

#### ONE-TO-ONE MATCHING, OR ONE-TO-ONE CORRESPONDENCE

Two sets can be compared by matching the members of one set with the members of the other set.

Let the set  $X = \{a, b, c, d\}$ , and let the set  $Y = \{p, q, r, s\}$ . The members of set  $X$  are not the same as the members of set  $Y$ , although both sets have the same number of members. To each member of set  $X$  there corresponds a member of set  $Y$ .

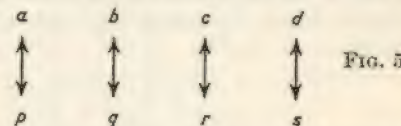


FIG. 5

Figure 5 shows that there is a one-to-one correspondence. When this occurs, the sets are said to be equivalent.

#### VENN DIAGRAMS OF SET DIAGRAMS

These are drawings which were invented by the mathematician John Venn in order to help one to understand the algebra of sets.

The universal set is usually represented by a rectangle, and it is called the set  $U$ . A subset is usually represented by a circle, and it is denoted by a capital letter other than  $U$ . Any number of subsets may be inserted in the rectangle and they may, or may not, overlap.

There are the following possibilities:

1. *Disjoint Sets*. Figure 6 shows the universal set  $U$  containing the subsets  $A$  and  $B$  which do not overlap. No member of set  $A$  is a member of set  $B$ , and vice versa.

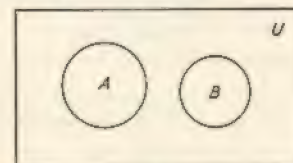


FIG. 6



*Example.*  $U$  is the set of all human beings.  
 $B$  is the set of all boys.  
 $G$  is the set of all girls.  
This is illustrated in Fig. 7.

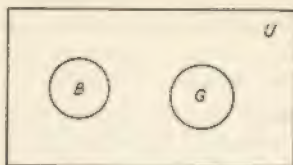


FIG. 7

2. *Intersecting Sets.*  $U$  is the set of all the pupils in a class who study French, Latin and German.

$F$  is the set of pupils who study French and German.

$L$  is the set of pupils who study Latin and German.

The shaded portion represents the pupils who study French, Latin and German (Fig. 8).

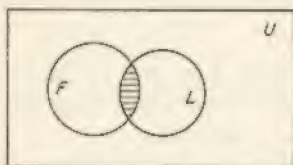


FIG. 8

3. *Subset of a subset.*

*Example.*  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ .

$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ .

$B = \{6, 7, 8\}$ .

$A$  is a subset of  $U$ , and  $B$  is a subset of the subset  $A$ . This is illustrated in Fig. 9.

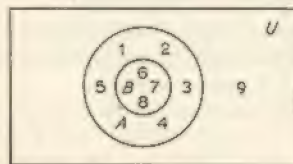


FIG. 9

4. *The Complement of a Set.* Let a set be represented by  $A$ . The complement of the set  $A$  is 'not  $A$ '. It contains those members not in  $A$ . It is denoted by  $A'$  (Fig. 10).

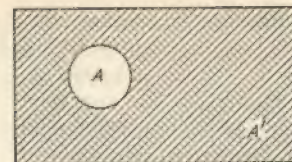


FIG. 10

# OPERATIONS WITH SETS

There are two main operations which can be performed with sets. These are:

- The union of sets, and denoted by the symbol  $\cup$  (read cup).
- The intersection of sets, and denoted by the symbol  $\cap$  (read cap).

1. *The Union Operation.* The union of two sets is another set formed by taking as its elements those elements that are in one or the other of the two sets being united. In a number system, the union operation would be the 'addition' operation.

*Example 1.* Let set  $A = \{1, 3, 5, 7\}$

and let set  $B = \{2, 4, 6, 8\}$ .

Then  $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ .

This is illustrated in Fig. 11.

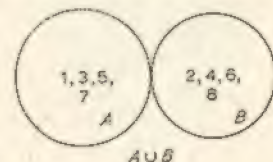


FIG. 11

*Example 2.* Let set  $P = \{a, b, c, d, e\}$

and let set  $Q = \{c, d, e, f\}$ .

Then  $P \cup Q = \{a, b, c, d, e, f\}$ .

This is illustrated in Fig. 12.

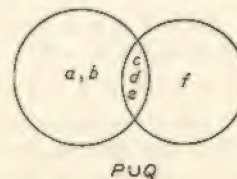


FIG. 12

Although the letters  $c$ ,  $d$ , and  $e$  occur in both sets they are not written twice.  $P \cup Q$  is the set which includes all the members of the sets on which the operation is performed.

2. *The Intersection Operation.* The intersection of two sets is another set formed by taking as its elements all those elements which are in both subsets being intersected.

In a number system, the intersection operation would be the 'multiplication' operation.

*Example 1.* Let set  $X = \{a, b, c, d, e\}$   
and let set  $Y = \{c, d, e, f\}$ .  
Then  $X \cap Y = \{c, d, e\}$ .

This is illustrated in Fig. 13 where the shaded portion represents  $X \cap Y$ .

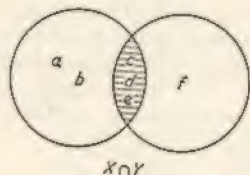


FIG. 13

The intersection process is performed when the H.C.F. of several numbers is being found in arithmetic.

*Example 2.* Let set  $A = \{2, 3, 5\}$   
and let set  $B = \{3, 5, 7\}$ .  
Then  $A \cap B = \{3, 5\}$ .

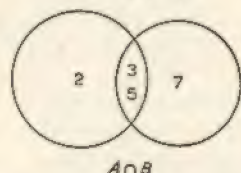


FIG. 14

To find the H.C.F. of 30 and 105, arrange the numbers in their factors:

$$\begin{aligned} 30 &= 2 \times 3 \times 5, \\ 105 &= 3 \times 5 \times 7, \end{aligned}$$

Therefore H.C.F. of 30 and 105 =  $3 \times 5 = 15$ .

*The complement of a Set.* Let  $U$  be the universal set, Fig. 15, and let  $A$  be a subset. Then the shaded portion represents the complement of  $A$ , and it is denoted by the symbol  $A'$ .

This means that the set  $A$  together with the complement of  $A$  (i.e.,  $A'$ ) make up the universal set. The complement of the set  $A$  could be described as 'not  $A$ '.

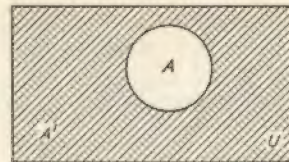


FIG. 15

*Example.* Let  $U$  represent the set of all the letters of the alphabet. Let  $A$  represent the set of all vowels. Then  $A'$  will represent the set of all the consonants.

*Examples on Sets.* Consider the following sets:

$$\begin{aligned} 1 &= \{x, y, z\}, & a &= \{y, z\}, & b &= \{z, x\}, & c &= \{x, y\}, \\ d &= \{x\}, & e &= \{y\}, & f &= \{z\}, & 0 &= \{ \}. \end{aligned}$$

Then:

1.  $a + b = a \cup b = \{x, y, z\} = 1$
2.  $c + e = c \cup e = \{x, y\} = c$
3.  $f + 1 = f \cup 1 = \{x, y, z\} = 1$
4.  $d + d = d \cup d = \{x\} = d$
5.  $a \times b = a \cap b = \{z\} = f$
6.  $c \times c = c \cap c = \{x, y\} = c$
7.  $c \times f = c \cap f = \{ \} = 0$
8.  $1 \times a = 1 \cap a = \{y, z\} = a$
9.  $a' = \{x\} = d$
10.  $b + e' = b \cup e' = \{x, z\} = b$ .



## CHAPTER SIX

### THE ALGEBRA OF SETS

#### 1. THE COMMUTATIVE LAW

This is the law concerning the order in which quantities are taken.

##### (a) For Addition, or Union

- (i) In arithmetic,  $3 + 4 = 4 + 3$
- (ii) In Algebra,  $a + b = b + a$
- (iii) In set theory,  $A \cup B = B \cup A$ .
- (iii) is illustrated thus:

Let set  $A = \{a, b, c, d\}$  (Fig. 16),  
 and let set  $B = \{c, d, e, f\}$ .  
 Then  $A \cup B = \{a, b, c, d, e, f\}$ ,  
 and  $B \cup A = \{a, b, c, d, e, f\}$ .  
 Therefore,  $A \cup B = B \cup A$ .

Therefore, the order in which sets are added is immaterial.

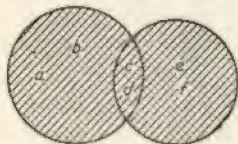


FIG. 16

##### (b) For Multiplication, or Intersection

- (i) In arithmetic,  $3 \times 4 = 4 \times 3$
- (ii) In algebra,  $a \times b = b \times a$
- (iii) In set theory,  $A \cap B = B \cap A$ .
- (iii) is illustrated thus:

Let set  $A = \{a, b, c, d\}$  (Fig. 17),  
 and let set  $B = \{c, d, e, f\}$ .  
 Then  $A \cap B = \{c, d\}$ , and  $B \cap A = \{c, d\}$ .  
 Therefore,  $A \cap B = B \cap A$ .

Therefore, the order in which sets are multiplied is immaterial.

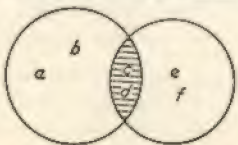


FIG. 17

#### 2. THE ASSOCIATIVE LAW

This is the law concerning the way in which quantities are grouped.

##### (a) For Addition, or Union

- (i) In arithmetic,  $(3 + 4) + 5 = 3 + (4 + 5)$
- (ii) In algebra,  $(a + b) + c = a + (b + c)$
- (iii) In set theory,  $(A \cup B) \cup C = A \cup (B \cup C)$ .
- (iii) may be illustrated thus:

Consider the following sets (see Fig. 18).

$A = \{a, b, c, d\}$ ,  $B = \{b, c, d, e, f\}$ ,  $C = \{c, d, e, g, h\}$ .

$$\begin{aligned} A \cup B &= \{a, b, c, d, e, f\} \\ (A \cup B) \cup C &= \{a, b, c, d, e, f, g, h\} \quad (i) \\ B \cup C &= \{b, c, d, e, f, g, h\} \\ A \cup (B \cup C) &= \{a, b, c, d, e, f, g, h\} \quad (ii) \end{aligned}$$

Comparing (i) and (ii),  $(A \cup B) \cup C = A \cup (B \cup C)$ .

Similarly, it may be shown that  $(A \cup B) \cup C = C \cup (A \cup B)$ .

Therefore, the order in which sets are added does not affect the result.

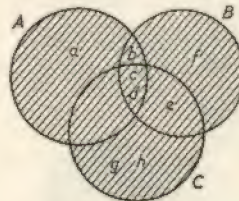


FIG. 18

##### (b) For Multiplication, or Intersection

- (i) In arithmetic,  $(3 \times 4) \times 5 = 3 \times (4 \times 5)$
- (ii) In algebra,  $(a \times b) \times c = a \times (b \times c)$
- (iii) In set theory,  $(A \cap B) \cap C = A \cap (B \cap C)$ .
- (iii) is illustrated as follows:

Consider the following sets (see Fig. 19).

$A = \{a, b, c, d\}$ ,  $B = \{b, c, d, e, f\}$ ,  $C = \{c, d, e, g, h\}$ .

$$\begin{aligned} A \cap B &= \{b, c, d\} \\ (A \cap B) \cap C &= \{b, c, d\} \cap \{c, d, e, g, h\} = \{c, d\} \\ B \cap C &= \{c, d, e\} \end{aligned}$$

$$A \cap (B \cap C) = \{a, b, c, d\} \cap \{c, d, e\} = \{c, d\}.$$

Therefore,  $(A \cap B) \cap C = A \cap (B \cap C)$ .

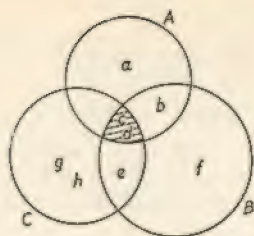


FIG. 19

### 3. THE DISTRIBUTIVE LAW

There are two distributive laws in the theory of sets.

*Law 1.* This is expressed symbolically as:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

In words, this is expressed as:

(a) To form the union of  $(B \cap C)$  with  $A$ ,  $B$  and  $C$  are 'cupped' individually with  $A$ , and the results are 'capped';

or

(b) The union of  $A$  and the intersection of  $B$  with  $C$  is equivalent to the intersection of the union of  $A$  and  $B$  with the union of  $A$  and  $C$ .

This is described by saying that 'union is distributive over intersection'. This is not true in ordinary algebra.

Law 1 is illustrated in Fig. 20.

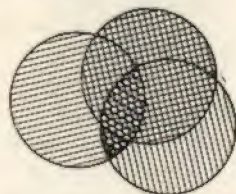


FIG. 20

*Law 2.* This is expressed symbolically as:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

In words, this is expressed as:

(a) To find the intersection of  $A$  with the union of  $B$  and  $C$ , obtain the union of the intersection of  $A$  with  $B$ , and  $A$  with  $C$ .

or

(b) To find the intersection of  $A$  with the union of  $B$  and  $C$ , 'cup' the 'cap' of  $A$  and  $B$  and the 'cap' of  $A$  and  $C$ .

In ordinary algebra,  $a \times (b + c) = a \times b + a \times c$ . Law 2 is illustrated in Fig. 21.

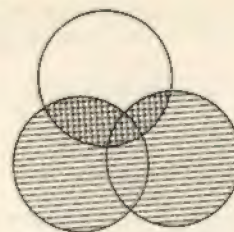


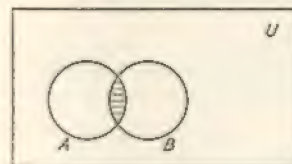
FIG. 21

### 4. DE MORGAN'S LAWS

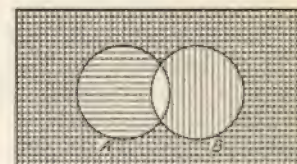
These laws deal with the complement of sets, and they have no counterpart in ordinary algebra. They are best illustrated by the use of Venn diagrams.

*Law 1.* This states:  $(A \cap B)' = A' \cup B'$ , i.e.  $(AB)' = A' + B'$ .

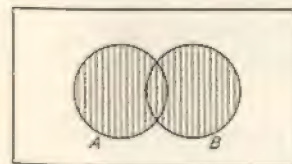
(i)  $(AB)' = (A \cap B)'$  is represented by the whole rectangle less the shaded portion (Fig. 22 (a)).



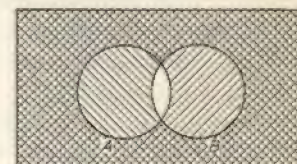
(a)



(b)



(c)



(d)

FIG. 22

(ii)  $A' + B' = A' \cup B'$  is represented by the portion unshaded (Fig. 22 (b)).

Therefore,  $(A \cap B)' = A' \cup B'$ , or  $(AB)' = A' + B'$ .

*Law 2.* This states:  $(A \cup B)' = A' \cap B'$ , i.e.  $(A + B)' = A'B'$ .



(iii)  $(A \cup B)' = (A + B)'$  is represented by the shaded portion (Fig. 22(c)).

(iv)  $A' \cap B' = A'B'$  is represented by the intersection of the shaded portions, which is the portion doubly shaded (Fig. 22(d)).

Therefore  $(A \cup B)' = A' \cap B'$ , i.e.  $(A + B)' = A'B'$ .

## 5. THE LAW OF ABSORPTION

This is best illustrated by means of a Venn diagram (Fig. 23).

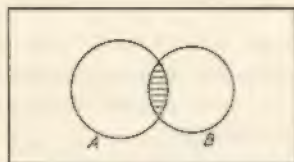


FIG. 23

It will be seen that the shaded portion represents  $A \times B$ , or  $A \cap B$ . This shaded portion is a subset of  $A$ . Therefore  $AB$  contains no element which is already a member of set  $A$ .

Therefore,  $A + AB = A$ .

This is the law of absorption. It can be used when a term appears in an expression which repeats another term together with some further factors.

*Example.*  $AC' + ABC' + AC' = AC'$ .

Here, the second and third terms repeat the first term with the additional factors  $B$  and  $1$  respectively. It is said that the second and third terms are absorbed in the first.

The law may also be illustrated thus:

Consider the expression  $a + ab$   
 Factorizing,  $= a(1 + b)$   
 Using  $1 + b = 1$ ,  $= a \times 1 = a$ .  
 Therefore,  $a + ab = a$ .

## THE LAWS OF BOOLEAN ALGEBRA

Set algebra is a Boolean algebra, and the laws are listed below, in which  $A, B, 1, 0$  have the following meanings:

$A + B$  is the union of  $A$  and  $B$ , or  $A \cup B$ ,  
 $AB$  is the intersection of  $A$  and  $B$ , or  $A \cap B$ ,

$1$  is the universal set, or  $U$ ,  
 $0$  is the empty set, or  $\emptyset$ .

Any number system which has the following properties constitutes a Boolean algebra:

1. Commutative laws.  $A + B = B + A$ .  $AB = BA$ .
2. Associative laws.  $A + (B + C) = (A + B) + C$ .  
 $A(BC) = (AB)C$ .
3. Distributive laws. (a)  $A(B + C) = AB + AC$ .  
 (b)  $A + BC = (A + B)(A + C)$ .
4. Union and intersection of a set with itself.

$$A + A = A. \quad AA = A.$$

5. Union of a set and its complement.  $A + A' = 1$ .
6. Intersection of a set and its complement.  $AA' = 0$ .
7. Complement of the universal set is the empty set.  $1' = 0$ .
8. Complement of the empty set is the universal set.  $0' = 1$ .
9. The union of a set and the universal set is the universal set.

$$1 + A = 1.$$

10. The intersection of a set and the empty set is the empty set.  
 $0 \times A = 0$ .
11. The union of a set and the empty set is the set.  $0 + A = A$ .
12. The intersection of a set and the universal set is the set.  
 $1 \times A = A$ .

## 13. De Morgan's Laws.

- (a) The complement of the union of  $A$  and  $B$  is the intersection of the complement of  $A$  and  $B$ .

$$(A + B)' = A'B'.$$

- (b) The complement of the intersection of  $A$  and  $B$  is the union of the complements of  $A$  and  $B$ .

$$(AB)' = A' + B'.$$

- (c) The complement of the complement of  $A$  is  $A$ .

$$(A')' = A.$$

## 14. The law of absorption. $A + AB = A$ .

## SUMMARY OF THE LAWS OF BOOLEAN ALGEBRA

- |                         |                              |
|-------------------------|------------------------------|
| 1. $a(b + c) = ab + ac$ | 2. $a + bc = (a + b)(a + c)$ |
| 3. $a + a = a$          | 4. $aa = a$                  |
| 5. $a + a' = 1$         | 6. $aa' = 0$                 |
| 7. $1' = 0$             | 8. $0' = 1$                  |

9.  $1 + a = 1$                       10.  $0 \times a = 0$   
 11.  $0 + a = a$                       12.  $1 \times a = a$   
 13.  $(a + b)' = a'b'$                 14.  $(ab)' = a' + b'$   
 15.  $(a')' = a$ .

Examples of the use of the above laws

1.  $x(x' + y) = xx' + xy = 0 + xy = xy$   
 2.  $c(c + d) = cc + cd = c + cd = c$   
 3.  $xy + xy' = x(y + y') = x \times 1 = x$   
 4.  $s(s' + t') = ss' + st' = 0 + st' = st'$   
 5.  $ab(a' + b') = aba' + abb' = 0 + 0 = 0$   
 6.  $yz'(y + z) = yz'y + yz'z = yz' + 0 = yz'$   
 7.  $ab(ab + a'c) = aabb + aba'c = ab + 0 = ab$   
 8.  $(p + q')(p' + q) = pp' + pq + p'q' + qy'$   
      $= 0 + pq + p'q' + 0 = pq + p'q'$   
 9.  $pqr + pqr' + pqs = pq(r + r') + pqs$   
      $= pq \times 1 + pqs = pq + pqs = pq$   
 10.  $(x + y')(xy + x'y') = xxy + xx'y' + y'xy + y'x'y'$   
      $= xy + 0 + 0 + x'y' = xy + x'y'$   
 11.  $(xy' + yz)(xz' + y'z') = xy'xz' + xy'y'z' + yzxz' + yzy'z'$   
      $= xy'z' + xy'z' + 0 + 0 = xy'z'.$

#### EXERCISE 7

1. Given the following sets:

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5\},$$

$$C = \{3, 4, 5, 6, 7\}, \quad D = \{6, 7, 8, 9\}.$$

Find:

1.  $A \cup B$                       2.  $A \cap B$                       3.  $A \cup D$   
 4.  $A \cap D$                       5.  $A \cup C \cup D$                 6.  $A \cap C \cap D$   
 7.  $A \cup (C \cap D)$             8.  $A \cap (C \cup D)$             9.  $(A \cup C) \cap (A \cup D).$

#### ANSWERS

1.  $\{1, 2, 3, 4, 5\}$                 2.  $\{3, 4\}$   
 3.  $\{1, 2, 3, 4, 6, 7, 8, 9\}$       4.  $\{ \}$   
 5.  $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$     6.  $\{ \}$   
 7.  $\{ \}$                               8.  $\{3, 4\}$   
 9.  $\{1, 2, 3, 4, 6, 7\}.$
2. Given the following sets:  
 $A = \{\text{Jack, John, Joan, Bill}\},$   
 $B = \{\text{Joan, Bill, Jim, Tony}\},$   
 $C = \{\text{Jack, Joan, Henry}\}.$

Find:

1.  $A \cap B$                               2.  $A \cup B$   
 3.  $(A \cap B) \cap C$                     4.  $B \cap C$   
 5.  $A \cap (B \cap C)$                     6.  $(A \cap B) \cup C.$

#### ANSWERS

1.  $\{\text{Joan, Bill}\}$                       2.  $\{\text{Jack, John, Joan, Bill, Tony}\}$   
 3.  $\{\text{Joan}\}$                               4.  $\{\text{Joan}\}$   
 5.  $\{\text{Joan}\}$                               6.  $\{\text{Jack, Joan, Bill, Henry}\}.$
3. Find the solution set of integers for the following:
1.  $\{x | x + 2 = x\}$                       2.  $\{x | 3x + 7 = 22\}$   
 3.  $\{x | 3x + 5 = 2x + 5 + x\}$         4.  $\{x | x^2 = 25\}$   
 5.  $\{x | x > 5\}$ , where  $U = \{1, 2, 3, 4, 5, 6, 7, 8\}.$

#### ANSWERS

1.  $\{ \}$                                       2.  $\{5\}$   
 3.  $\{\text{all numbers}\}$                     4.  $\{+5, -5\}$   
 5.  $\{6, 7, 8\}.$
4. List the members of the following sets:
1. The set of even numbers between 10 and 20 divisible by 3.  
 2. The set of months of the year with names beginning with M.  
 3. The set of days of the week with names beginning with T.  
 4. The set of all odd numbers between 7 and 9.  
 5. The set of vowels in the word 'arithmetic'.  
 6. The set of all natural numbers between 6 and 12.

#### ANSWERS

1.  $\{12, 18\}$                               2.  $\{\text{March, May}\}$   
 3.  $\{\text{Tuesday, Thursday}\}$             4.  $\{ \}$   
 5.  $\{a, e, i\}$                               6.  $\{7, 8, 9, 10, 11\}.$
5. Given the following sets:  
 $A = \{a, b, c, d, e\}, P = \{p, q, r\}, N = \{1, 2, 3\},$   
 state whether the following are true or false:
1.  $a \in P$                                   2.  $2 \in P$                                   3.  $a \in N$   
 4.  $b \in A$                                   5.  $q \notin P$                                   6.  $0 \in N$   
 7.  $r \in P$                                   8.  $3 \in N$                                   9.  $d \in A.$

#### ANSWERS

1. False                                      2. False                                      3. False  
 4. True                                        5. False                                      6. False  
 7. True                                        8. True                                        9. True.



6. If  $x \in U$ , and  $U$  is the set of all the natural numbers, make a list of the following subsets of  $U$ :

1.  $\{x | x + 2 > 5\}$
2.  $\{x | 3x - 1 = 5\}$
3.  $\{x | 5x - 1 = 2\}$
4.  $\{x | \frac{1}{2}x = 5\}$
5.  $\{x | x + 6 = 6 + x\}$ .

ANSWERS

1.  $\{4, 5, 6, \dots\}$
2.  $\{2\}$
3.  $\emptyset$
4.  $\{10\}$
5.  $U$ .

## CHAPTER SEVEN

### THE GEOMETRY OF SETS

Although there is no mathematical definition of a point, everyone has a good idea of what a point is. It has no size or shape, but merely position. However, a set of points can constitute a line. A line is an infinite set of points, and any point on the line is a member of the set. It is not possible to draw a straight line, but only a segment of the line. It is possible to show that there are as many points on a line one inch long as there are points on a line two inches long.

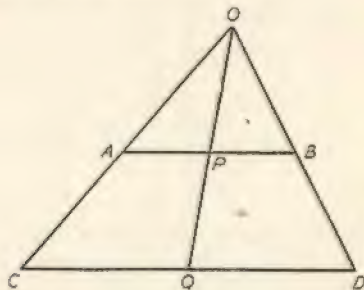


Fig. 24

Let  $AB$  (Fig. 24) be a line one inch long, and let  $CD$  be a line two inches long. Let  $O$  be any point in the plane of  $AB$  and  $CD$ .  $P$  is any point on  $AB$ .  $OP$  produced meets  $CD$  in  $Q$ . For every point such as  $P$  on  $AB$  there is a point such as  $Q$  on  $CD$ . Also, for every point such as  $Q$  on  $CD$  there is a point such as  $P$  on  $AB$ . This is an example of a one-to-one correspondence. On the two lines  $AB$  and  $CD$ , there are no points without a corresponding point on the other.

Not only is a line a set of points; a plane is also a set of points. A plane has only two dimensions, length and breadth. It has no thickness.

Let  $O$  (Fig. 25) be a point in a plane. Draw, in the plane, a number of points which are a constant distance from  $O$ . These points will constitute members of the set of points which are equidistant from  $O$ . The totality of this infinite set of points will constitute a circle. This circle is called the locus of points equidistant from  $O$ .



FIG. 25

In Fig. 26,  $X$  and  $Y$  are two fixed points. The set of points which are equidistant from  $X$  and  $Y$  constitute the perpendicular bisector of  $XY$ .

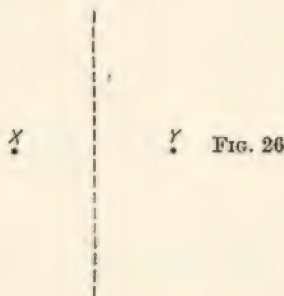


FIG. 26

Figure 27 shows two intersecting straight lines  $PQ$  and  $RS$ . The set of points which are equidistant from the lines  $PQ$  and  $RS$  constitute the bisectors of the angles between  $PQ$  and  $RS$ . Actually there are two sets of points to be considered, the set marked  $l$ , and the set marked  $m$ .

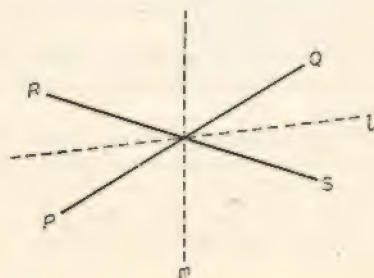


FIG. 27

## UNION AND INTERSECTION OF LINES

1. Let it be required to draw a circle to pass through the three given points  $X, Y, Z$  (Fig. 28). Let the centre of this circle be  $O$ . Then  $O$  must be a member of the set of points equidistant from  $X$  and  $Y$ , and  $O$  must also be a member of the set of points equidistant from  $Y$  and  $Z$ . Call these sets the sets  $L$  and  $M$ , as indicated in the Figure. Then  $O \in L$  and  $O \in M$ .

Therefore,  $O$  is a member of the intersection of  $L$  and  $M$ , i.e.,

$$O \in L \cap M.$$

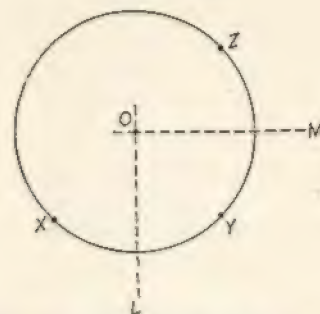


FIG. 28

2. Let it be required to draw a circle to touch externally the three sides of a triangle. Let the triangle be  $ABC$  (Fig. 29). Produce  $AC$  to  $D$  and  $AB$  to  $E$ . Then the circle will touch the lines,  $CD, CB, BE$ . Its centre will be on the lines bisecting the angles  $BCD$  and  $CBE$ . Let the bisecting lines be  $L$  and  $M$  as indicated in the Figure. Then  $O \in L$  and  $O \in M$ . Therefore  $O$  is a member of the intersection of  $L$  and  $M$ .

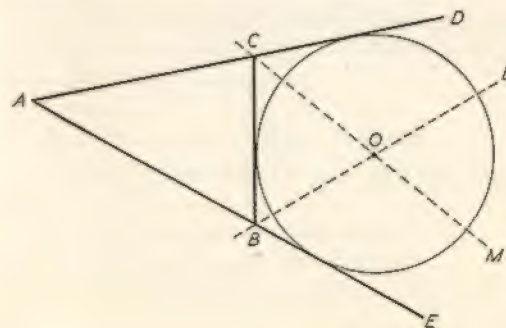


FIG. 29



Therefore,

$$O \in L \cap M.$$

If  $AD$  is represented by  $D$  and  $BC$  by  $B$ , then  $M \in B \cup D$ .

If  $BE$  is represented by  $E$ , then  $L \in B \cup E$ .

Therefore,  $O \in (B \cup D) \cap (B \cup E)$ .

## CHAPTER EIGHT

### GROUPS

Let a binary operation be denoted by the symbol  $*$ , and let a set of elements be called  $G$ . Then the set  $G$  will form a group if the following rules, called axioms, are obeyed.

1. If the sets  $X$  and  $Y$  belong to  $G$ , then  $X * Y$  is an element of  $G$ . This means that  $G$  is closed under the operation. In other words, the combination of two elements gives another element which is a member of the set  $G$ .
2. The operation  $*$  is associative, so that

$$X * (Y * Z) = (X * Y) * Z.$$

This means that the order of combination of any three elements does not affect the result.

3.  $G$  must contain an identity element named  $I$ , such that

$$X * I = I * X = X, \text{ for all } X \text{ in } G.$$

This means that one element has the property that, if it be combined with any other element, the element remains unchanged.

4. For every  $X$  in  $G$  there must exist a unique element  $X'$ , known as an inverse element, such that

$$X * X' = X' * X = I.$$

This means that every element of the set  $G$  possesses an inverse.

A group which obeys the commutative law is called a commutative, or Abelian, group. This means that  $X * Y = Y * X$  for every pair of elements.

#### THE GROUP STRUCTURE OF THE INTEGERS

The following considerations show that the integers form a group:

- (a) The operation of addition is associative
- (b) The system contains a zero element
- (c) For every integer  $n$ , its negative  $-n$  is also in the system.

## RINGS

A system of elements is called a ring if there are two binary operations defined for the system having the following properties:

1. Both operations are associative.
2. The system is an Abelian group with respect to one of the operations. This operation is denoted by the sign  $+$ , and is called addition.
3. The other operation is distributive with respect to addition. If this other operation is called multiplication and is represented by  $\cdot$ , then the distributive law takes the form:

$$(a) \ a \cdot (b + c) = a \cdot b + a \cdot c,$$

and

$$(b) \ (b + c) \cdot a = b \cdot a + c \cdot a.$$

The distributive law must be stated in two parts (a) and (b) because the operation of multiplication need not be commutative.

It will be seen from the above properties that the system of integers forms a ring.

## FIELDS

A field is defined as a ring in which a unity element exists, and which has a reciprocal for every element except zero. It has a double group structure, one for addition and one for subtraction.

## CHAPTER NINE

### TRANSFORMATIONS OR MOTION GEOMETRY

A common property of the transformations to be considered here is the conservation of straightness. This means that straight lines in the 'object' will become straight lines in the 'image'. There will also be conservation of distance. Thus the 'image' will be the same shape and size as the 'object', and the 'object' and its 'image' will be congruent figures. Distance-preserving properties are referred to as isometries, which means 'equal measures'. Such transformations are sometimes called Rigid Motions. Since two translations, when performed in succession, can be equivalent to a third translation, translations can form a closed system, and so constitute a group.

#### REFLECTION

When an object is placed in front of a mirror, an image is formed behind the mirror, so that object and image are of the same shape and at the same distance from the mirror. There is lateral inversion, however, the image being back to front, and left to right instead of right to left.

Let the straight line  $PQ$  (Fig. 30) be reflected in the mirror  $XY$ , and let the image be  $P'Q'$ .  $P$  and  $P'$  are equidistant from  $XY$ . Also,  $P'Q'$  is the same length as  $PQ$ , and  $PQ$  and  $P'Q'$  are equally inclined to  $XY$ .

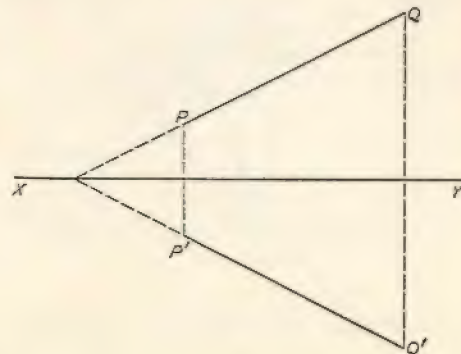


FIG. 30



(a) *Reflection in the Coordinate Axes.* Let the point  $P_1 = (x, y)$  be reflected in the axes (Fig. 31). Then the images will be  $P_2 = (-x, y)$ ,  $P_3 = (-x, -y)$ ,  $P_4 = (x, -y)$ .

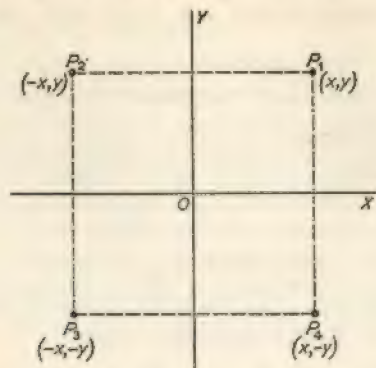


FIG. 31

(b) *Reflection in the Line  $y = x$ .* This is illustrated in Fig. 32. The point  $P_1$  is reflected in the line  $y = x$ , and produces an image at  $P_2$  such that  $P_1M = P_2M$ . The triangles  $OP_1M$  and  $OP_2M$  will be congruent, so that  $OP_1 = OP_2$ , and angle  $P_1OM = \text{angle } P_2OM$ .

Therefore triangles  $OP_1A$  and  $OP_2B$  will be congruent.

Therefore  $OB = OA = x$  and  $BP_2 = AP_1$ .

Therefore  $P_2$  will be the point  $(y, x)$ .

Therefore, on reflection in the line  $y = x$ , the coordinates of a point are interchanged.

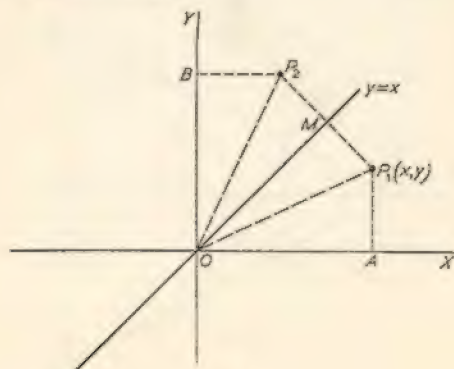


FIG. 32

(c) *Reflection in the Line  $x + y = 0$ .* It may be shown that in this case the point  $(x_1, y_1)$  becomes the point  $(-y_1, -x_1)$ . Thus the coordinates will be interchanged and the signs will be changed.

As a reflection reverses the sense of irregular shapes, no translation or rotation or combination of them can have the same effect, although both a translation and a rotation can be produced by successive reflections. Thus, reflections may be considered as basic isometries, because all other isometries can be obtained by suitable combinations of reflections.

#### ROTATION

This is a common isometry and occurs wherever there are wheels. Every line of a rotating figure turns through the same angle. In order to specify a rotation it is necessary to state the centre about which all points are rotated, the angle of rotation, and the direction of rotation. It will be considered here that all points will remain in the same plane. The centre of rotation will be the only fixed point. The convention adopted in geometry is that an anticlockwise turn is positive. However, bearings are measured positively from the North in a clockwise direction.

An important rotation is the half turn, a rotation through  $180^\circ$ , or  $\pi$  radians. This causes the signs of coordinates to be reversed (Fig. 33).

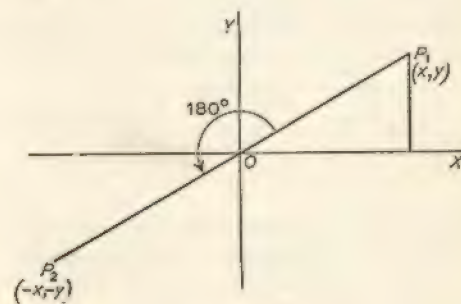


FIG. 33

Figure 34 illustrates a quarter turn, and transforms the point  $P_1 = (x, y)$  into the point  $P_4 = (-y, x)$ .

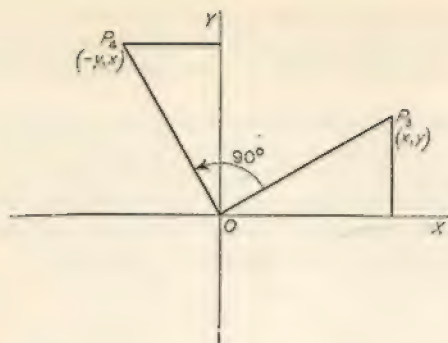


FIG. 34

#### ROTATIONAL SYMMETRY

This kind of symmetry is frequently met in nature. Figure 35 shows the mystic pentagram.  $P_1, P_2, P_3, P_4, P_5$  are equidistantly placed around the circumference of a circle. They are produced by five anticlockwise rotations of  $\frac{360^\circ}{5} = 72^\circ$  around the centre  $O$ . The points are carried on to themselves by five reflections in the lines joining  $O$  to each point.

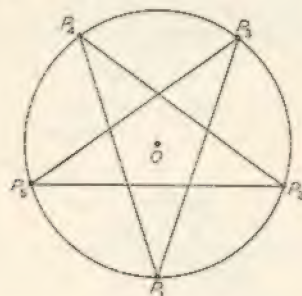


FIG. 35

The ten operations of reflection and rotation form a group, the group of self-transformations of the pentagram.

The best known examples of rotational symmetry are snowflakes which possess hexagonal symmetry.

#### TRANSLATION

In this kind of transformation all points in the plane are moved equal distances in the same direction. The effect in any

particular figure is such that it remains the same way up. It could in some respects act as an introduction to vectors. A good example of translation could be the flight of a squadron of aircraft in formation. Figure 36 shows the effect of a translation on the line  $P_1Q_1$ . It becomes the line  $P_2Q_2$ . All points on the line are moved the same distances parallel to each axis. Let the  $x$  coordinate be moved a distance  $h$ , and let the  $y$  coordinate be moved a distance  $k$ . Then each point  $(x, y)$  on the line  $P_1Q_1$  will become the point  $(x + h, y + k)$  on the line  $P_2Q_2$ .

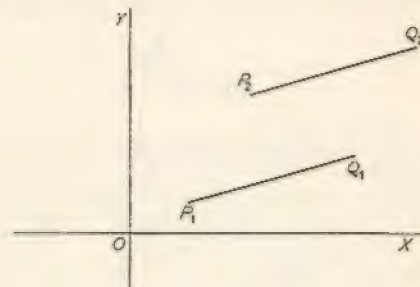


FIG. 36

#### SHEARING

Shear translation is a basic transformation in that group of transformations called affinities. If a reversible linear mapping is followed by a translation, the product is an affine transformation. The system of such products is an affine group. In a shear translation each point moves parallel to a given line and the distance moved by each point is proportional to the distance

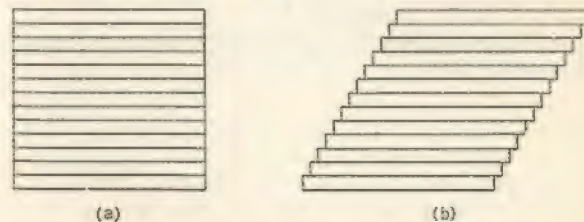


FIG. 37

of the point from the line. Shearing transformation conserves the area, or the volume, of the figure transformed. This can be seen in the case of a pile of books (Fig. 37). If the pile is



subjected to a shearing force the shape is changed from that shown in Fig. 37(a) to that shown in Fig. 37(b). It will be seen that the total area of the end is the same, and the total volume of books is unaltered. However, shearing is not a shape-preserving transformation.

#### SIMILARITY

Similarity, or dilatation, or enlargement is a transformation which preserves shape. That is, shape is invariant. This is the kind of transformation obtained when a photograph is 'enlarged'. This invariant property of similar right-angled triangles

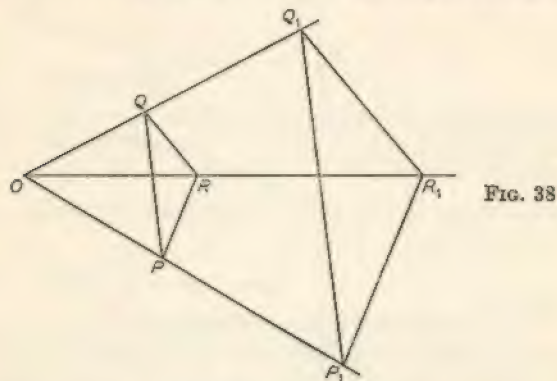


FIG. 38

is used in trigonometry. A dilatation may be defined as a transformation in which a point  $P$  and its transform  $P_1$  are in line with a fixed point  $O$ , such that  $OP_1 = \lambda \cdot OP$ ,  $\lambda$  being a constant, called the dilatation constant.

Figure 38 shows triangle  $PQR$  transformed under dilatation into triangle  $P_1Q_1R_1$ .

## CHAPTER TEN

### TOPOLOGY

This is sometimes called the rubber sheet geometry. It is the study of the properties of shapes which are invariant under a topological transformation. All the shapes in Fig. 39 are the

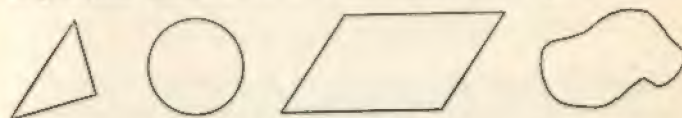


FIG. 39

same topologically, the invariant property which they all possess is that they all have an inside and an outside. If each shape were drawn on a sheet of rubber, any one could, by suitable stretching, be transformed into any other. One condition is that there should be no tearing of the rubber. This means that any two points which are neighbours before the transformation will remain so afterwards, and any two points not neighbours before will not be neighbours after the transformation. The shapes are said to be topologically equivalent.

The shapes of Fig. 40 are also topologically equivalent.

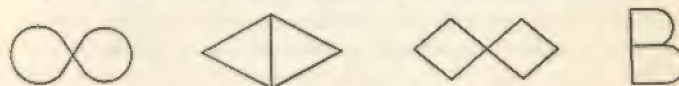


FIG. 40

In the same way, the solid figures, sphere, cone, cylinder, and pyramid, are all topologically equivalent. They all have an inside and an outside.

Just like ordinary geometry, topology concerns lines, points, and figures, except that in topology they are allowed to change their size and their shape. Topology deals with position rather than with size and shape. It deals with the properties of figures which remain the same however the figures are stretched or bent. There is no meaning for distance in topology. If two points are one foot apart, this could be stretched to two feet, or three feet and so on. Similarly, the straightness, or curvature, of the line has no meaning; neither has the size of angles.

All these can be changed without affecting a problem. Thus, in topology, there are no rigid bodies. Lines can stretch or bend, and plane and solid figures can change their shape. However, lines, and plane and solid figures are continuous. A line has no holes or gaps.

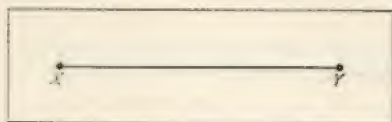


FIG. 41

Figure 41 shows two points  $X$  and  $Y$ . They are joined by a line  $XY$  called the arc  $XY$ . The line  $XY$  may be made curved and longer, but providing it is not made to cross itself it is always the path from  $X$  to  $Y$ .

The shapes of Fig. 39, namely the triangle, circle, quadrilateral, and irregular closed curves are all topologically the same. Each is called a closed circuit, or simple closed curve.

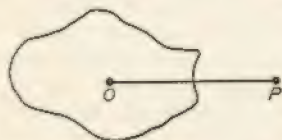


FIG. 42

Figure 42 shows just one of these figures; but they can all be treated alike.  $O$  is a point outside, and  $P$  is a point inside. However the figure be distorted, the line  $OP$  will always cross an arc of the curve. This is because there are no gaps in the contour line: it is a result of the principle of continuity. Thus a closed curve will divide the plane into two parts, an inside and an outside. Since it is always necessary to cross the boundary line in order to pass from the inside to the outside, the action is said to be invariant. A situation which is invariant is one which does not change. Although it is possible to change the shape of a line or a figure by bending or stretching, these form topological transformations, but they do not alter the topological nature of the line or figure. In order to maintain the topological nature of a line or figure it is necessary that it be neither cut, torn, nor folded. If a closed curve is cut or torn or folded a new topological figure is formed. Thus a line or curve has another topological invariant, and that is the order of the

points on it. If  $P$ ,  $Q$ ,  $R$ , and  $S$  are consecutive points on a line or figure, then they will remain in that order under a topological transformation. A topological transformation always produces equivalent figures, so that a circle, square, quadrilateral, ellipse, triangle are all equivalent. If the ends of a line are joined, or if a closed curve is cut, new figures are formed.

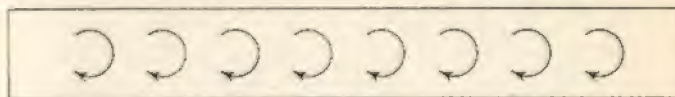


FIG. 43

*Example.* Figure 43 shows a strip of paper which is marked on both sides with a continuous pattern of oriented circles. Form this into (a) a cylinder and (b) a Moebius band (see p. 51). Is it possible to find a point on either band at which the sense of rotation of two adjacent circles suddenly changes? It will be possible to find such a point  $M$  on the Moebius band, and the surface is said to be non-orientable. There will be no such point on the cylinder, which is said to have an orientable surface.

#### CLASSIFICATION OF TOPOLOGICAL FIGURES

A sphere, a cube, and a pyramid are topologically equivalent. They are continuous surfaces, and divide space into two regions, an inside and an outside. They are called simple closed surfaces. Each can be changed into the other by simple distortion.

In topology, surfaces or figures are classified by the number of cuts needed to simplify the surface or figure. Figure 44 shows

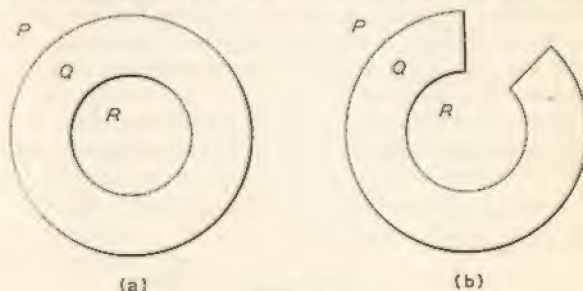


FIG. 44

a disc-like figure with a hole, similar to an annulus. It divides the plane into three regions,  $P$ ,  $Q$ , and  $R$ . If Fig. 44(a) is cut



as shown in Fig. 44(b), a simple closed curve is obtained, in which  $P$  and  $R$  are both on the outside.

Three dimensional figures are classified in the same way, by finding how many cuts are needed to change the object into a simpler closed surface like a sphere.

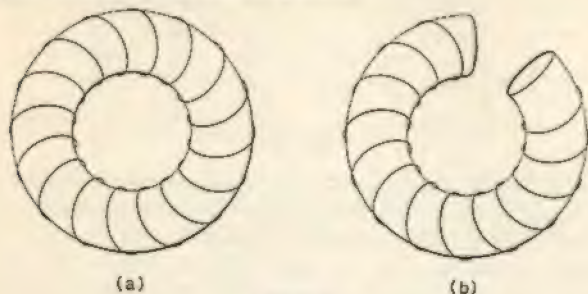


FIG. 45

Figure 45 shows a torus, a doughnut-shaped solid. A single cut will produce Fig. 45(b), which is a simple closed surface. As a torus and a sphere are topologically different, they are said to be different in connectivity. The hole in the torus is actually outside the body, not inside it.

A sheet of paper is said to have two surfaces and one edge. A cylinder which is open at both ends has two surfaces and two edges. The inner tube of a motor car wheel, like a torus, has two surfaces but no edges. It is possible to classify shapes by counting the cross-cuts needed without dividing it into more than one piece. A cross-cut should begin and end on an edge.

If a sheet of paper is cut from edge to edge two sheets are formed, but if an open cylinder is so cut, only one sheet is produced because it can then be opened up into a single flat sheet. This means that a sheet of paper is a simple surface, and the cylinder is a singly connected surface. A sphere is also a singly connected surface because it is possible, although difficult, to make a hole in the surface and stretch it into a flat sheet.

#### EULER'S FORMULA

This is a formula concerning regular polyhedra. A polyhedron is a solid whose faces consist of a number of polygons. The polygons may be a triangle, a quadrilateral, a pentagon, or a hexagon. Figure 46 shows regular polyhedra. All the faces are congruent figures, and all the angles at the vertices are equal.

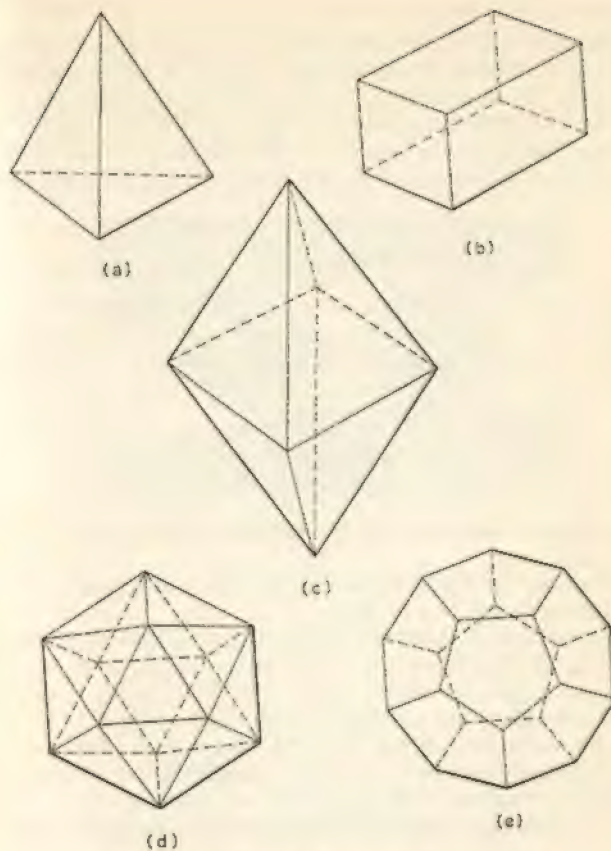


FIG. 46. (a) Tetrahedron (b) Hexahedron  
(c) Octahedron (d) Icosahedron (e) Dodecahedron

In a simple polyhedron, let the number of vertices =  $V$ , the number of edges =  $E$ , and the number of faces =  $F$ . Then Euler's formula states:

$$V - E + F = 2.$$

This is a topological formula, because it concerns only the numbers of the vertices, edges, and faces, and not the sizes of them, or the lengths, areas, or straightness of the parts.

## EULER'S NETWORKS

Leonard Euler, the Swiss mathematician, was a pioneer in the study of networks. His researches into these problems were started by the Königsburg bridge problem. Königsburg was the name of a Russian town. A river flowed through this town, and two islands in the middle of the river were joined to the banks by seven bridges, as shown in Fig. 47. The problem was,

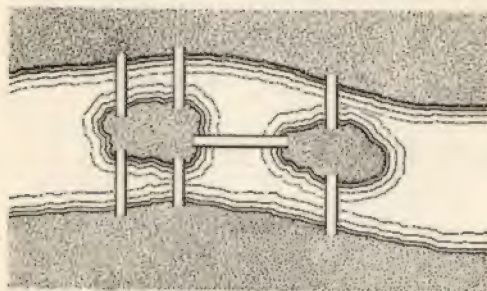


FIG. 47

"Is it possible to cross each of the bridges, and cross each bridge once only?"

Euler discovered that it was not possible to cross each bridge without recrossing any of them.

Euler tackled the problem by drawing a network  $P, Q, R, S$ , as in Fig. 48. In this network, the vertices are  $P, Q, R$ , and  $S$ . These are points where the lines cross. The joining lines are called axes. Since the number of arcs at the vertex  $P$  is 3,  $P$  is then an odd vertex.  $Q$  has five arcs, and so it is also an odd vertex. Euler discovered the following rule: If the number of vertices at which an odd number of arcs meet is two or less, then it is possible to draw the figure in one continuous pencil

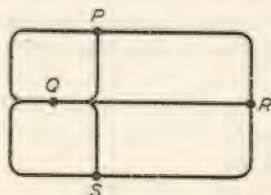


FIG. 48

movement without drawing any part more than once. If there are more than two vertices at which an odd number of arcs meet, then such a drawing is not possible.

Euler also discovered that any network with only even vertices could be traversed in one journey. He also discovered that if a network contained two and only two odd vertices it could be traversed in one journey, but it would not be possible to return to the starting point. It would be necessary to start at one odd vertex and finish at the other odd vertex. He also discovered that if a network contained four or a higher even number of odd vertices it would not be possible to traverse the network in one journey.

## THE MOEBIUS STRIP

In the eighteenth century the mathematician Moebius discovered that there were surfaces with only one side. The simplest is the Moebius band. This is just a rectangular strip of paper with the two ends joined together after making a half twist (Fig. 49).

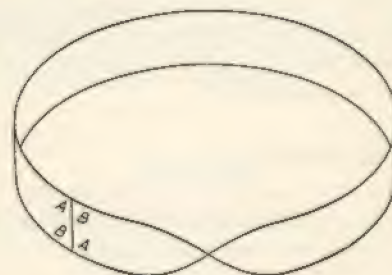
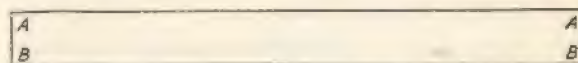


FIG. 49

If an insect crawled along the middle line of this strip, it would return to its original position upside down, and it could walk from any point to any other point, on either side, without crossing an edge. If a painter started to paint the outside of the band, he would paint the inside as well.

## THE KLEIN BOTTLE

This is a one-sided surface named after the German mathematician Felix Klein. It is a single surface with no inside,



outside, or edges (Fig. 50). It is formed by drawing the smaller end of a tapering tube through one side of the tube and then enlarging the former until it fits the latter.

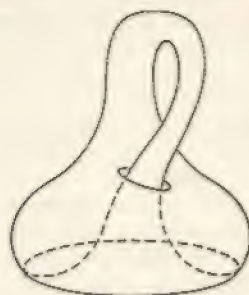


FIG. 50

#### THE MAP COLOUR PROBLEM

This is a famous unsolved problem in mathematics. Maps are often drawn with countries, which have a common border, coloured differently. It has been found, so far, that only four different colours are needed for this purpose. No map has yet been conceived whose colouring requires more than four colours. It has never been proved that no more than four different colours will ever be necessary.

#### THREE-TO-THREE CONNECTIONS

(a) Three houses  $P$ ,  $Q$ , and  $R$  are to be supplied with water, gas and electricity. A local authority requirement is that pipes and wires must not cross each other (Fig. 51). Experiment

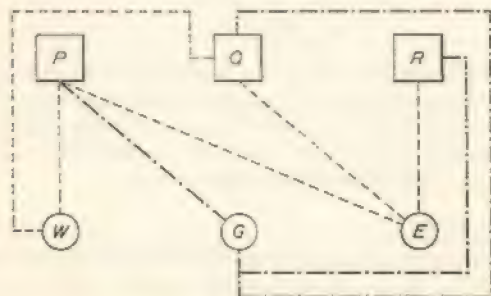


FIG. 51

has shown that it is not possible to connect  $P$ ,  $Q$ , and  $R$  with  $W$ ,  $G$ , and  $E$  and still fulfil the authority's requirement.

(b) This is the problem of the Caliph's daughters. They had so many suitors that the Caliph set each one a problem, which was

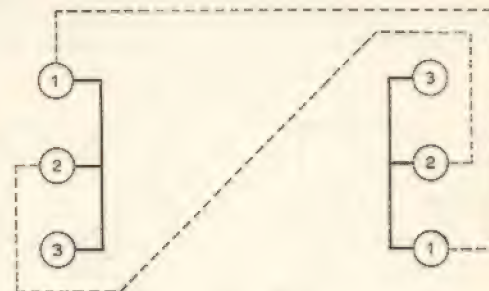


FIG. 52

to connect the two sets of figures, 1, 2, 3, shown in Fig. 52, so that the same figures are joined without the connecting lines crossing each other or any lines in the Figure. This is a topological problem, and it has no solution.

#### EXERCISE 8

1. State which of the diagrams in Fig. 53 are topological lines.

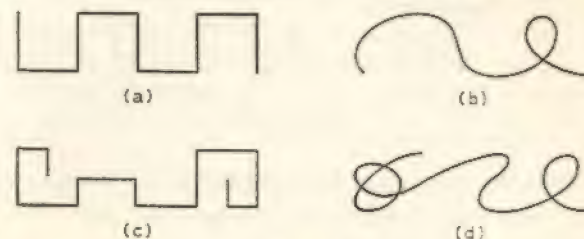


FIG. 53

2. State which of the diagrams shown in Fig. 54 are simple closed curves.

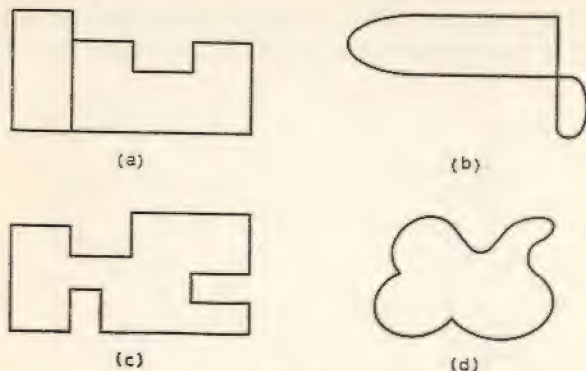


FIG. 54

3. For each of the networks given in Fig. 55, state the number of even and odd vertices, and investigate whether the network can be traversed.

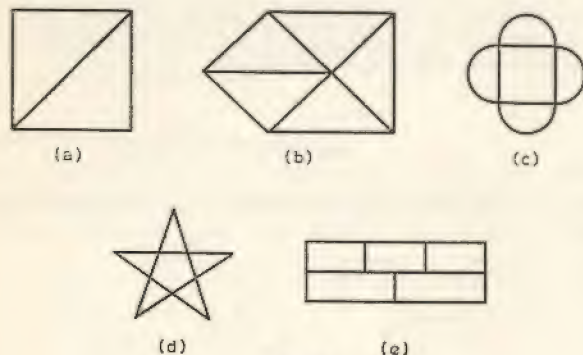


FIG. 55

4. Classify the following surfaces regarding their connectivity:

1. a tennis ball
2. a hosepipe
3. an inner tube
4. a coat
5. a plate.

5. Make a Moebius band which contains two half-twists. Mark it and cut it along the middle. Find out whether the surface is

- (a) one sided or two sided and (b) orientable or non-orientable.
6. Show that the least number of intersections possible with four houses and three supply stations is three.
7. An area of tessellated flooring is completely covered with regular hexagons. How many colours are necessary to ensure that adjacent hexagons are of different colour?
8. Find the minimum number of trips needed to trace this network given in Fig. 56.

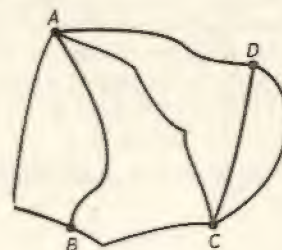


FIG. 56

#### ANSWERS TO EXERCISE 8

1. (a), (c)
2. (c), (d)
- 3.

Network	Even vertices	Odd vertices	Traversable
(a)	2	2	yes
(b)	0	6	no
(c)	4	0	yes
(d)	10	0	yes
(e)	4	8	no

4. (1) singly connected (2) singly connected  
(3) doubly connected (4) doubly connected  
(5) simply connected
5. (a) two sided, (b) orientable
7. Three
8. One.



# CHAPTER ELEVEN

## THE ALGEBRA OF CIRCUITS

The algebra of logic has a direct application to the design of electrical circuits. However, in that which follows, no mention will be made of voltage, current, power, or resistance. The chapter will be concerned only with the result of inserting switches into circuits. Sections of circuits will be denoted by the small letters,  $a$ ,  $b$ ,  $c$ , etc. If section  $a$  has a current flowing through it, this fact will be expressed as

$$a = 1.$$

If there is no current flowing through the section  $a$ , then this will be expressed as

$$a = 0.$$

Diagrammatically, these situations are illustrated in Fig. 57.

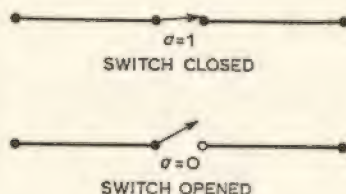


FIG. 57

It must be noted that the 0 and the 1 are not numbers in these situations. 0 and 1 are the only symbols needed, because a switch can be either open or closed, and nothing else.

(i) *Switches in Series.* Figure 58 shows two switches in series.



FIG. 58

A current will flow only when both switches are closed. This situation is denoted by  $ab$ .

(ii) *Switches in Parallel.* In this situation (Fig. 59), current will



FIG. 59

flow when either  $a$  or  $b$  is closed, or both are closed. This is represented by  $a + b$ .

### TABLES

(iii) *The Multiplication Table.* This follows from (i) above.

$a$	$b$	$ab$
0	0	0
0	1	0
1	0	0
1	1	1

(iv) *The Addition Table.* This follows from (ii) above.

$a$	$b$	$a + b$
0	0	0
0	1	1
1	0	1
1	1	1

(v) *Series Parallel Circuit (a).* Figure 60 shows circuit  $a$  in series

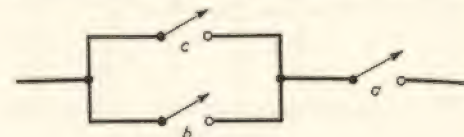


FIG. 60

with  $b$  and  $c$  in parallel. This is represented by  $a(b + c)$ . The following table shows the result of various combinations of switches open or closed.

$a$	$b$	$c$	$a(b + c)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(vi) *Series Parallel Circuit (b)*. Figure 61 shows series circuit  $a$  and  $b$  in parallel with series circuit  $a$  and  $c$ . The whole circuit will be represented by  $ab + ac$ .

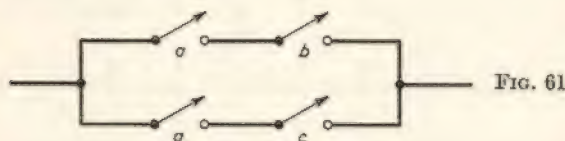


FIG. 61

The following table shows the result of various combinations of switches open or closed.

$a$	$b$	$c$	$ab + ac$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

#### TWO-WAY SWITCHES

Some switches are arranged in such a way that in one position current flows through one circuit and in the other position current flows through another circuit. This is illustrated in Fig. 62.

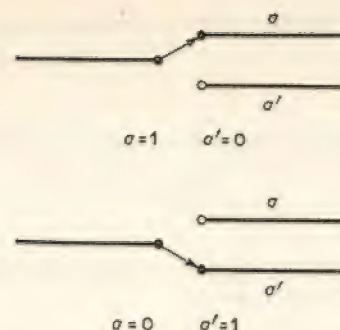


FIG. 62

It will be seen that there is no intermediate position.  $a'$  means that the state of wire  $a'$  is opposite to the state of wire  $a$ .  $a'$  means that the switch is open when  $a$  is closed and closed when  $a$  is open.

*Example.* The diagrammatic representation of  $a + a'$  is shown in Fig. 63(a) or (b). There are two possibilities.

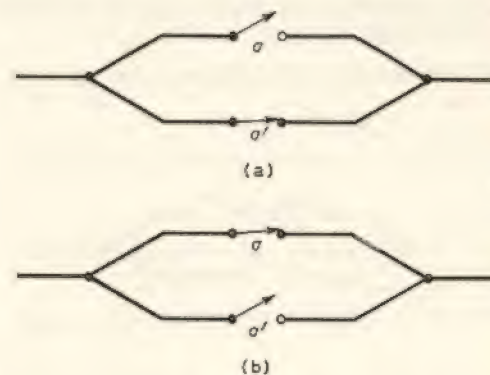


FIG. 63

The addition table for this arrangement is:

$a$	$a'$	$a + a'$
0	1	1
1	0	1



*Example.* The symbolic representation of the circuit in Fig. 64 is  $aa' = 0$ .



FIG. 64

*Example.* The symbolic representation of the circuit in Fig. 65 is  $(a + b)(a + c) = a + bc$ .

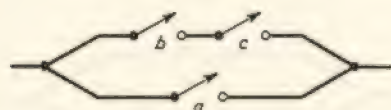


FIG. 65

*Example.* The diagrammatic representation of the formula  $a + bc$  is given in Fig. 66(a) or (b).

There are two equivalent circuits since

$$a + bc = (a + b)(a + c).$$



$$a + bc = (a + b)(a + c) \\ \text{(a)}$$



$$a + bc = (a + b)(a + c) \\ \text{(b)}$$

FIG. 66

The table for this circuit is

$a$	$b$	$c$	$a + bc$
1	1	1	1
1	1	0	1
1	0	1	1
0	1	1	1
1	0	0	1
0	1	0	0
0	0	1	0
0	0	0	0

*Example.* The circuit of Fig. 67 may be represented by the formula  $(ab + c)(ab + c')$ .



FIG. 67

$$\text{However, } (ab + c)(ab + c') = ab + cc' \text{ (Distributive law)} \\ = ab + 0 = ab.$$

Therefore the circuit of Fig. 67 may be simplified to the circuit of Fig. 68.

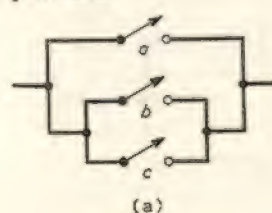


FIG. 68

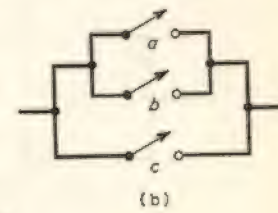
*Example.* By the associative law for addition,

$$a + (b + c) = (a + b) + c.$$

This means that the circuits in Fig. 69(a) and (b) are equivalent.



(a)



(b)

FIG. 69

It will be seen from the foregoing that the laws of Boolean algebra can be applied to switching circuits, in the same way as the laws can be applied to sets. A structure which behaves in this way is said to be polyvalent. Boolean algebra can be used in the simplification of electronic circuits, as will now be shown.

Consider the circuit indicated in Fig. 70.

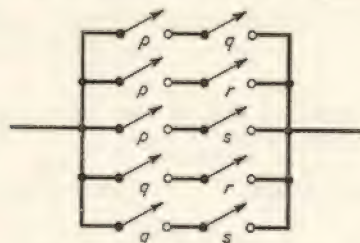


FIG. 70

This circuit consists of five parallel circuits, each made up of two switches in series. It will be seen that this can be expressed in Boolean algebra by the formula

$$pq + pr + ps + qr + qs.$$

The expression may be factorized into

$$p(q + r + s) + q(r + s).$$

This expression represents two parallel circuits

$$(1) p(q + r + s) \text{ and } (2) q(r + s).$$

(1) contains  $p$  in series with  $q + r + s$  in parallel and (2) contains  $q$  in series with  $r + s$  in parallel. Therefore it can be represented diagrammatically as in Fig. 71.

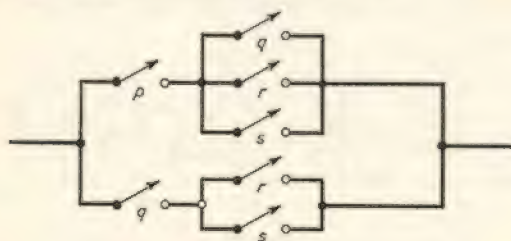


FIG. 71

This circuit (Fig. 71) will have the same switching effect as the circuit in Fig. 70.

However, there is another arrangement.

Since

$$pq + pr + ps + qr + qs = pq + (p + q)(r + s),$$

the original circuit may be arranged according to the expression  $pq + (p + q)(r + s)$  which is indicated in Fig. 72.



FIG. 72

### EXERCISE 9

1. Write down the Boolean function for the circuits shown in Fig. 73, simplifying the expressions where possible.

2. Expand the following expressions and simplify where possible:

$$(i) a(a + b) \quad (ii) ab'(a + b) \quad (iii) (a + b')(a' + b)$$

3. Factorize and simplify the following:

$$(i) ab + ac \quad (ii) a' + a'b \quad (iii) ab + bc + bc' \\ (iv) ab + c \quad (v) ab + a'b'$$

### ANSWERS TO EXERCISE 9

$$1. (i) a.a = a \quad (ii) a + a = a \quad (iii) a + ab = a \\ (iv) (a + b)(a + c) = a + bc \\ (v) (a' + b')(a' + b)(a + b) = a'b \\ (vi) (a + a)(a + b)(a + c) = a \\ (vii) a + b(a' + c) = a + b \\ (viii) ab + a'(b + c')(b' + c)$$

$$2. (i) a + ab = a \quad (ii) ab' \quad (iii) ab + a'b'$$

$$3. (i) a(b + c) \quad (ii) a'(1 + b) = a' \quad (iii) b \\ (iv) (a + c)(b + c) \quad (v) (a + b')(a' + b).$$



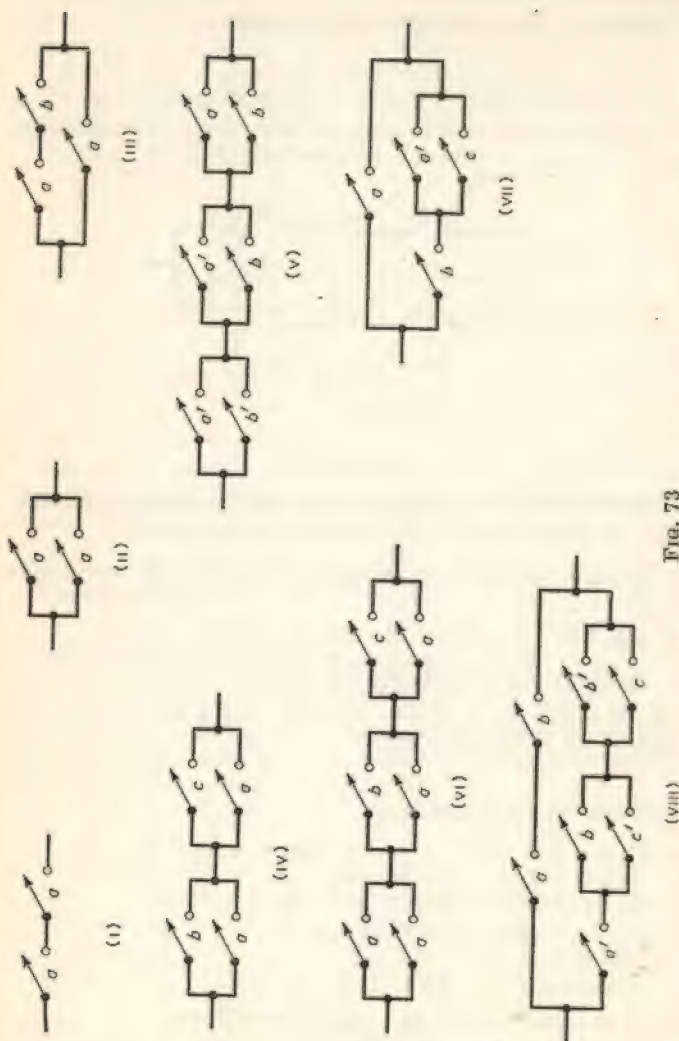


Fig. 73

## CHAPTER TWELVE DETERMINANTS

A determinant is an arrangement of letters or numbers set out in equal numbers of rows and columns, and the arrangement is placed between two vertical straight lines. Here are some examples:

$$\begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix}, \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \begin{vmatrix} 3 & 2 & 7 \\ 4 & 6 & 9 \\ 2 & 8 & 5 \end{vmatrix}.$$

A determinant with two rows and two columns is called a determinant of the second order, and one with three rows and three columns is a determinant of the third order.

The symbol  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$  stands for  $ad - bc$

The symbol  $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$  stands for

$$a \begin{vmatrix} q & r \\ y & z \end{vmatrix} - b \begin{vmatrix} p & r \\ x & z \end{vmatrix} + c \begin{vmatrix} p & q \\ x & y \end{vmatrix}$$

$\begin{vmatrix} q & r \\ y & z \end{vmatrix}$  is called the minor of  $a$ ,

$\begin{vmatrix} p & r \\ x & z \end{vmatrix}$  is called the minor of  $b$ , and

$\begin{vmatrix} p & q \\ x & y \end{vmatrix}$  is called the minor of  $c$ .

The co-factor of an element is the minor of that element with the proper sign attached.

The sign of a co-factor is positive or negative according as the sum of the number of the row and the number of the column struck out to form the minor is even or odd.

*Example*  $\begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} = 3 \times 2 - 4 \times 5 = 6 - 20 = -14.$

*Example*  $\begin{vmatrix} 3 & 2 & 7 \\ 4 & 6 & 9 \\ 2 & 8 & 5 \end{vmatrix} = 3 \begin{vmatrix} 6 & 9 \\ 8 & 5 \end{vmatrix} - 2 \begin{vmatrix} 4 & 9 \\ 2 & 5 \end{vmatrix} + 7 \begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix}$

$$= 3(6 \times 5 - 8 \times 9) - 2(4 \times 5 - 2 \times 9) + 7(4 \times 8 - 2 \times 6)$$

$$= 3 \times -42 - 2 \times 2 + 7 \times 20$$

$$= -126 - 4 + 140 = 10.$$

#### RULE OF SARRUS

This is a rule for finding the value of a determinant.

The first two columns of the determinant are repeated on the right of the determinant, as shown in Fig. 74. The expansion of

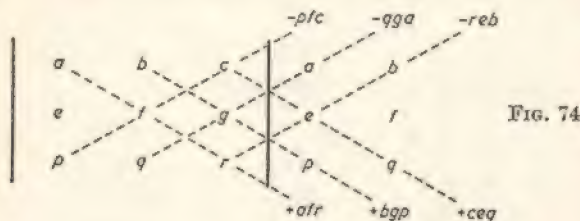


FIG. 74

the determinant is written down by taking the algebraic sum of the products formed by the elements on each of the six diagonals shown, the products taken downwards being positive and those taken upwards being negative. The determinant may also be evaluated by repeating the first two rows below the determinant and multiplying as before.

*Example.* Use the rule of Sarrus to evaluate  $\begin{vmatrix} 3 & 2 & 7 \\ 4 & 6 & 9 \\ 2 & 8 & 5 \end{vmatrix}$

as shown in Fig. 75.

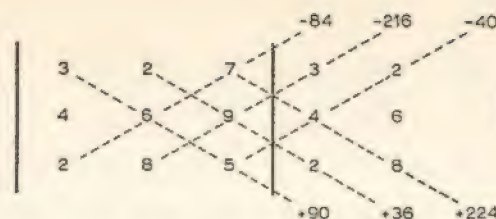


FIG. 75

$$\text{Value} = 90 + 36 + 224 - 84 - 216 - 40$$

$$= 350 - 340 = 10.$$

#### SOLUTION OF EQUATIONS BY DETERMINANTS

Consider the equations  $a_1x + b_1y = c_1$   
 $a_2x + b_2y = c_2.$

It may be shown that their solution is given by

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_1 \end{vmatrix}}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_1 \end{vmatrix}}$$

Thus the roots of the two simultaneous equations may be expressed in determinant form.

*Example.* Solve the equations  $8x + y = 3$   
 $9x + 2y = -1.$

The solutions are:

$$x = \frac{\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 8 & 1 \\ 9 & 2 \end{vmatrix}} = \frac{6 + 1}{16 - 9} = \frac{7}{7} = 1$$



$$y = \frac{\begin{vmatrix} 8 & 3 \\ 9 & -1 \\ 8 & 1 \\ 9 & 2 \end{vmatrix}}{\begin{vmatrix} -8 & -27 \\ 16 & -9 \end{vmatrix}} = \frac{-35}{7} = -5.$$

If there are three equations in three unknowns, such as:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The solution is

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

The denominator, in each case, is a third order determinant, and it is usually denoted by the symbol  $\Delta$ .

$$\text{Then } x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta},$$

in which,

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

#### PROPERTIES OF DETERMINANTS

1. The value of a determinant is unaltered if rows are changed to columns

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. If two rows (or two columns) are interchanged, the sign of the determinant is changed.

$$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

3. If two rows (or two columns) of a determinant are the same, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

4. If the elements of any row (or any column) of a determinant be each multiplied by the same factor, the result is the product of that factor and the original determinant.

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

5. If each element in any row (or any column) consists of the sum of two terms, the determinant can be expressed as the sum of two determinants.

$$\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$$

6. If the elements of any row (or any column) are increased or decreased by equal multiples of the corresponding elements of one or more of the other rows (or columns), the value of the determinant is unaltered.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix}$$

#### EXERCISE 10

Evaluate the following determinants:

$$1. \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \quad 2. \begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix} \quad 3. \begin{vmatrix} -6 & -5 \\ 3 & 8 \end{vmatrix}$$

$$4. \begin{vmatrix} 2 & 5 & 4 \\ 7 & 3 & 6 \\ 5 & 8 & 9 \end{vmatrix} \quad 5. \begin{vmatrix} 5 & 1 & -2 \\ 4 & 6 & 3 \\ 3 & 8 & 4 \end{vmatrix} \quad 6. \begin{vmatrix} 12 & 3 & 8 \\ 16 & 7 & 2 \\ 4 & -5 & 1 \end{vmatrix}$$

$$7. \begin{vmatrix} 2 & -2 & -1 \\ 6 & 1 & -1 \\ 4 & 3 & 5 \end{vmatrix} \quad 8. \begin{vmatrix} 4 & -3 & 2 \\ 5 & 9 & -7 \\ 4 & -1 & 4 \end{vmatrix} \quad 9. \begin{vmatrix} 3 & -4 & -3 \\ 2 & 7 & -31 \\ 5 & -9 & 2 \end{vmatrix}$$

Use determinants to solve the following equations:

10.  $2x + y = 4, \quad 3x + 4y = 1$
11.  $5x + 3y = -6, \quad 3x + 5y = -18$
12.  $5x + 2y = 2, \quad 3x - 5y = 26$
13.  $3x - 2y = 6, \quad 5x - y = -4$
14.  $2x - 5y = 10, \quad 14x + 5y = 6$
15.  $4x - 2y + z = 7, \quad 3x + 7y - 3z = 8, \quad 5x - y + 2z = 5$
16.  $2x - 3y + 5z = 4, \quad 3x + 2y + 2z = 3, \quad 4x + y - 4z = -6$
17.  $2x + 3y + z = 6, \quad 2x + 9y + 3z = 14, \quad 4x + y + 2z = 7$
18.  $7x - 4y + 2z = 4, \quad 2x + 3y - 7z = -5, \quad 5x - 2y - z = -1.$

#### ANSWERS TO EXERCISES 10

1. -2    2. 11    3. -33    4. -43    5. -35
6. -684    7. 70    8. 178    9. 0
10.  $x = 3, y = -2$     11.  $x = \frac{3}{2}, y = -\frac{3}{2}$
12.  $x = 2, y = -4$     13.  $x = -2, y = -6$
14.  $x = 1, y = -\frac{8}{5}$     15.  $x = 2, y = -1, z = -3$
16.  $x = -\frac{1}{3}, y = \frac{2}{3}, z = \frac{3}{5}$     17.  $x = 1, y = 1, z = 1$
18.  $x = 2, y = 4, z = 3.$



# CHAPTER THIRTEEN

## MATRICES

A matrix is a rectangular arrangement, or array, of elements. The elements may, or may not, be numbers. A matrix may be regarded as a code of instructions. It can have any number of rows and any number of columns. When describing a matrix, the number of rows is placed first. Thus a  $3 \times 4$  matrix will contain three rows and four columns. Matrices are arrangements such as:

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 5 \\ 1 & -4 & 6 \\ 7 & 2 & -1 \end{bmatrix}, [a \ b \ c], \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The elements are enclosed in brackets to distinguish them from determinants; it is possible for a matrix to have only one element. Matrices have their uses in many branches of mathematics; in particular, they are useful in the study of reflection, translation, rotation and other distortions of geometric figures. A matrix has no numerical value, because its sole purpose is to indicate an arrangement of elements, although the elements could be numbers. A square matrix is an arrangement having an equal number of rows and columns. Such a matrix must not be regarded as a determinant, which is a number obtained by multiplying, adding, and subtracting the elements in a certain manner. Although rows and columns can be interchanged in a determinant, they cannot be interchanged in a matrix without altering it. If a matrix has only one row, it is known as a row matrix. A matrix which has only one column is called a column matrix. To save space in printing, a column matrix is often written within braces as  $\{x_1, x_2, x_3, x_4\}$ . A single number could be regarded as a  $1 \times 1$  matrix, that is, as a matrix with only one element. Also to save space, a matrix is sometimes denoted by a capital letter, or a small letter in Clarendon type.

## MULTIPLICATION OF MATRICES

(a) Let  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$  be two matrices. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$$

(b) Two matrices can only be multiplied if the number of rows in one is equal to the number of columns in the other, they are then said to be compatible.

(i) Let  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $\begin{bmatrix} p \\ q \end{bmatrix}$  be two matrices. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} ap + bq \\ cp + dq \end{bmatrix}$$

(ii) Let  $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$  and  $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$  be two matrices. Then

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \times \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \\ ep + fr & eq + fs \end{bmatrix}$$

(iii) Let  $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$  and  $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$  be two matrices. Then

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x_1b_1 + x_2b_2 + x_3b_3 \\ y_1b_1 + y_2b_2 + y_3b_3 \end{bmatrix}$$

## ADDITION OF MATRICES

Matrices are added by adding corresponding elements.

Thus

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} + \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} a_1 + x_1 & b_1 + y_1 & c_1 + z_1 \\ a_2 + x_2 & b_2 + y_2 & c_2 + z_2 \end{bmatrix}$$

## SUBTRACTION OF MATRICES

Matrices are subtracted by subtracting corresponding elements.

Thus

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} - \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} a_1 - x_1 & b_1 - y_1 & c_1 - z_1 \\ a_2 - x_2 & b_2 - y_2 & c_2 - z_2 \end{bmatrix}$$

## MULTIPLYING A MATRIX BY A FACTOR

This is defined by the following:

$$K \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix}$$

## EQUAL MATRICES

Two matrices with numbers as elements are said to be equal if, and only if, they have the same number of elements arranged in exactly the same pattern and have the same numbers in the same places.

The sign  $\Rightarrow$  means 'implies' and the sign  $\Leftrightarrow$  means 'implies and is implied by'. Therefore:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Leftrightarrow a = p, b = q, c = r, d = s.$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} p & q & r \\ s & t & u \end{bmatrix} \Leftrightarrow a = p, b = q, c = r, d = s, e = t, f = u.$$

## THE UNIT MATRIX

This is denoted by I and then

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

When the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  multiplies another matrix, the

second matrix remains unchanged. The matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , there-

fore, behaves like the natural number 1. That is why it is called the unit matrix. It is also sometimes called the neutral matrix or the identity matrix. It can have other forms, depending

upon the order. Thus I may also be the matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,

which is a  $3 \times 3$  matrix. If  $a$  is a square matrix and I is a unit matrix of the same order, then  $I.a = a.I = a$ .

## THE ZERO MATRIX

Any matrix whose elements are all zero is called a zero matrix. The following are all zero matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The zero matrix is sometimes denoted by 0.

If  $A$  is the matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then,  $A.0 = 0.A = 0$ .

The zero matrix is sometimes called the null matrix.

## A DIAGONAL MATRIX

This is a square matrix whose elements are all zero except

those in the leading diagonal. For example,  $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$  is a

diagonal matrix.

A unit matrix is thus a diagonal matrix.



## THE INVERSE MATRIX

Let one matrix be  $A$ , and let a second matrix be  $B$ . If  $A.B = I$ , then the matrix  $B$  is the inverse of matrix  $A$ . Therefore,  $A.B = B.A = I$ .

If  $B$  is written as  $A^{-1}$ ,  $A^{-1} = A^{-1}.A = I$ .

### METHOD OF FINDING THE INVERSE OF A $2 \times 2$ MATRIX

Let  $A$  be the matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Then the inverse of  $A = A^{-1} = \begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$ ,

where  $\Delta = ad - bc$ , and called the determinant of the matrix.

*Example.* Find the inverse of  $A = \begin{bmatrix} 7 & 2 \\ 5 & 3 \end{bmatrix}$ .

Here,  $\Delta = 3 \times 7 - 5 \times 2 = 21 - 10 = 11$ .

Now, interchange the 3 and the 7, change the signs of the 2 and the 5, and divide each element by 11.

Then  $A^{-1} = \begin{bmatrix} \frac{3}{11} & \frac{-2}{11} \\ \frac{-5}{11} & \frac{7}{11} \end{bmatrix}$ .

*Example.* Find the inverse of  $A = \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}$ .

Here,  $\Delta = 2 \times 9 - 5 \times 3 = 18 - 15 = 3$ .

Interchange the 2 and the 9, change the signs of the 5 and the 3, and divide each element by  $\Delta = 3$ .

Then  $A^{-1} = \begin{bmatrix} \frac{9}{3} & \frac{-3}{3} \\ \frac{-5}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -\frac{5}{3} & \frac{2}{3} \end{bmatrix}$ .

### SIMULTANEOUS EQUATIONS USING MATRICES

Consider the equations  $a_1x + b_1y = c_1$   
 $a_2x + b_2y = c_2$ .

These may be written symbolically as:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Let the matrix  $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$  be called  $A$ , so its inverse will be  $A^{-1}$ .

Then  $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ .

Multiplying throughout by  $A^{-1}$

$$A^{-1}.A \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

From this,  $x$  and  $y$  may be determined.

*Example.* Solve the equations  $3x + y = 5$   
 $5x + 2y = 8$ .

These may be written in matrix form as:

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \dots \dots \dots (1)$$

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}. \text{ Therefore, } A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

Multiply (1) throughout by  $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ .

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

But 
$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \times 5 - 1 \times 8 \\ -5 \times 5 + 3 \times 8 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -25 & +24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$\therefore x = 2, y = -1.$

*Example.* Solve the equations  $10x + 4y = 12$   
 $7x + 3y = 8.$

These may be written as  $\begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} \quad (1)$

$A = \begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix}.$  Then  $A^{-1} = \begin{bmatrix} \frac{3}{2} & -4 \\ -7 & \frac{10}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -2 \\ -7 & 5 \end{bmatrix}.$

Multiplying both sides of (1) by  $A^{-1}$

$$\begin{bmatrix} \frac{3}{2} & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix}.$$

But 
$$\begin{bmatrix} \frac{3}{2} & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \times 12 + (-2) \times 8 \\ -7 \times 12 + 5 \times 8 \end{bmatrix} = \begin{bmatrix} 18 + (-16) \\ -42 + 40 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

$\therefore x = 2, y = -2.$

#### EXERCISE 11

Use matrices to solve the following equations:

1.  $x + 3y = 5$                       2.  $5x + 2y = 11$   
 $2x + y = 9$                         3.  $x + y = 6$

3.  $4x + y = 9$                       4.  $x - 3y = 2$   
 $3x + 4y = 10$                       5.  $x + y = 26$
5.  $x + 2y = 10$                       6.  $x + 3y = 1$   
 $x + y = 6$                           3.  $x + y = -5.$

#### ANSWERS TO EXERCISE 11

1. 2, 1                                  2. 1, 3                                  3. 2, 1
4. 5, 1                                  5. 2, 4                                  6. -2, 1.

#### APPLICATIONS OF MATRICES

1. An anticlockwise rotation of  $90^\circ$  about the origin.

Let the point  $(x, y)$  be transformed to the point  $(x_1, y_1)$  (Fig. 76). Since triangles  $P_0OM_1$  and  $P_1OM_2$  are congruent, then

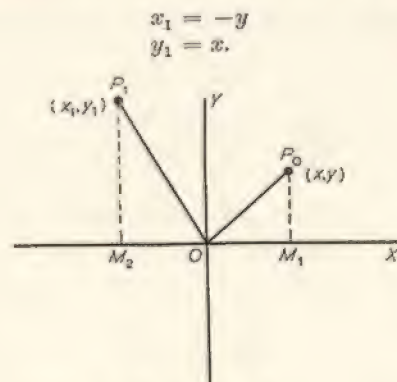


FIG. 76

These equations may be written as

$$x_1 = 0 \cdot x - 1 \cdot y$$

$$y_1 = 1 \cdot x + 0 \cdot y.$$

These may be written in matrix form as

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore the matrix describing this particular transformation is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$



## 2. Reflection in the $x$ -axis.

Let the point  $(x, y)$  be transformed under reflection in the  $x$ -axis to the point  $(x_1, y_1)$  (Fig. 77).

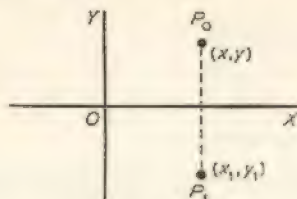


FIG. 77

It will be seen that  $x_1 = x$

$$y_1 = -y.$$

These equations may be written as

$$x_1 = 1.x + 0.y$$

$$y_1 = 0.x - 1.y.$$

These may be written in matrix form as

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore the matrix describing this particular transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

## 3. Enlargement.

Let the vector  $OP_0$  be stretched to become the vector  $OP_1$ , such that  $OP_1 = K.OP_0$  (Fig. 78).

Then

$$x_1 = Kx$$

and

$$y_1 = Ky.$$

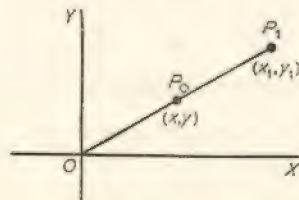


FIG. 78

These equations may be expressed as

$$x_1 = Kx + 0.y$$

$$y_1 = 0.x + K.y,$$

or, expressing them in matrix form,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore, the matrix describing the transformation of enlargement is

$$\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$$

## 4. Rotation through an angle $\theta$ .

Let the point  $P_0$  move to the point  $P_1$ , so that the vector  $OP_0$  rotates through an angle  $\theta$ . Let  $OP_0 = OP_1 = r$  (Fig. 79).

Then  $r \cos(\alpha + \theta) = r \cos \alpha \cos \theta - r \sin \alpha \sin \theta$ .

$$\therefore x_1 = x \cos \theta - y \sin \theta.$$

and  $r \sin(\alpha + \theta) = r \sin \alpha \cos \theta + r \cos \alpha \sin \theta$ .

$$\therefore y_1 = y \cos \theta + x \sin \theta.$$

This gives the following equations,

$$x_1 = \cos \theta . x - \sin \theta . y$$

$$y_1 = \sin \theta . x + \cos \theta . y.$$

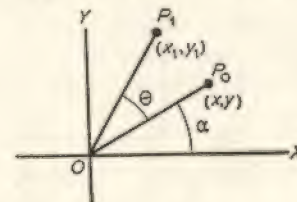


FIG. 79

These may be written in matrix form as

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the matrix describing the transformation is

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

# MULTIPLICATION OF MATRICES

Let  $A$  and  $B$  stand for two geometrical operations, and let  $AB$  represent the matrix which represents the result of performing the operations  $B$  and  $A$ , in turn.

Let operation  $B$  send the point  $P_0$  to  $P_1$ .

This may be expressed  $P_1 = BP_0$ .

Let the operation  $A$  send the point  $P_1$  to  $P_2$ .

This may be expressed  $P_2 = AP_1$ .

If these two results are combined, it will be possible to write

$$P_2 = ABP_0.$$

## ROTATION THROUGH $180^\circ$

*Method 1.* Let  $P_0$  be the point  $(x, y)$  and let  $P_1$  be the point  $(x_1, y_1)$ , separated from  $P_0$  by  $90^\circ$  (Fig. 80).

Let  $P_2$  be the point  $(x_2, y_2)$ , separated from  $P_0$  by  $180^\circ$ . It will be seen that the following equations hold:

$$x_2 = -x, \text{ and } y_2 = -y.$$

Expressed in matrix form, these give

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \dots \quad (1)$$

Thus the matrix describing a rotation of  $180^\circ$  is expressed

$$\text{in matrix form as } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

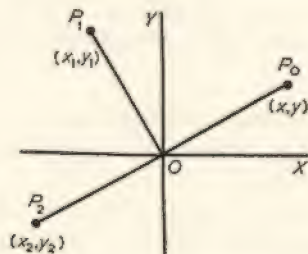


FIG. 80

*Method 2.*  $P_0$  may have been transformed to  $P_2$  by way of the point  $P_1$ .

Thus:

(a) Transform  $P_0$  to  $P_1$  and obtain

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(b) Transform  $P_1$  to  $P_2$  and obtain

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

(c) Now combine (a) and (b) and obtain

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \dots \quad (2)$$

By comparing (1) and (2) it follows that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This gives the result of multiplying together two matrices.

The rule for the multiplication of matrices may be found in the following manner.

$$\text{Let } \begin{matrix} x_1 = ax + by & x_2 = ax_1 + by_1 \\ y_1 = \gamma x + \delta y & y_2 = cx_1 + dy_1. \end{matrix}$$

By eliminating  $x_1$  and  $y_1$  from these equations, the following results will be obtained.

$$\begin{matrix} x_2 = (ax + by)x + (a\beta + b\delta)y \\ y_2 = (cx + dy)x + (c\beta + d\delta)y. \end{matrix}$$

If these equations are written in matrix form, they become

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} ax + b\gamma & a\beta + b\delta \\ cx + d\gamma & c\beta + d\delta \end{bmatrix}.$$

This equation may be simplified to:

$$A \cdot B = C.$$

The above gives a rule for the multiplication of two matrices.

$$\text{Example. Multiply } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$



The rule for the multiplication of matrices may be expressed in words as follows. Let  $A$  be an  $l \times m$  matrix (with  $l$  rows and  $m$  columns), and let  $B$  be an  $m \times n$  matrix (with  $m$  rows and  $n$  columns). Then the product matrix,  $AB (= C)$  is an  $l \times n$  matrix such that if  $C_{rs}$  is the element in the  $r^{\text{th}}$  row and  $s^{\text{th}}$  column of  $C$ , it is the sum of the products of elements in the  $r^{\text{th}}$  row of  $A$  with the corresponding elements in the  $s^{\text{th}}$  column of  $B$ .

This may be expressed symbolically as,

$$C_{rs} = \sum_{t=1}^m a_{rt}b_{ts}.$$

It is necessary that the number of columns of  $A$  must equal the number of rows of  $B$ , or the product will not exist.

#### ADDITION OF MATRICES

The addition of matrices is defined for matrices of the same order. The sum of two matrices is the matrix which is obtained by adding together corresponding elements. Subtraction of matrices is defined by the subtraction of corresponding elements.

*Example 1.*

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+1 & 6+2 \\ 7+3 & 8+4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

*Example 2.*

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5-1 & 6-2 \\ 7-3 & 8-4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

#### EXERCISE 12

1. Form the matrices  $A.B$  and  $B.A$  given that

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}.$$

2. Form the matrices  $A.B$  and  $B.A$  given that

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

3. Multiply the following matrices

$$(i) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad (ii) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$4. \text{ Find } \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 6 \\ 3 & 2 \end{bmatrix}.$$

$$5. \text{ If } A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -2 \\ 0 & 5 \end{bmatrix}, \text{ find the matrices}$$

- (i)  $A + B$ , (ii)  $2A$ , (iii)  $A - B$ .

6. Use matrices to solve  $3x + y = 6$ ,  $5x + 2y = 11$ .

7. Find the effect of the matrix  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  on the square which

has its vertices at  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ ,  $(0, 1)$ .

8. Given that  $x_3 = x_2 + y_2$ ,  $x_2 = x_1 + 2y_1$ ,  $x_1 = 2x_0 - y_0$  and  $y_3 = x_2 - y_2$ ,  $y_2 = 2x_1 - y_1$ ,  $y_1 = x_0 + 2y_0$ , express  $x_3$  and  $y_3$  in terms of  $x_0$  and  $y_0$ .

#### ANSWERS TO EXERCISE 12

$$1. \begin{bmatrix} 10 & 7 \\ 12 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 7 \\ 4 & 17 \end{bmatrix} \quad 2. \begin{bmatrix} 4 & 3 \\ 7 & 4 \end{bmatrix}, \begin{bmatrix} 8 & 1 \\ 5 & 0 \end{bmatrix}.$$

$$3. (i) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (ii) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad 4. \begin{bmatrix} 1 & 8 \\ 7 & -1 \end{bmatrix}.$$

$$5. (i) \begin{bmatrix} 5 & 1 \\ 4 & 7 \end{bmatrix}, \quad (ii) \begin{bmatrix} 2 & 6 \\ 8 & 4 \end{bmatrix}, \quad (iii) \begin{bmatrix} -3 & 5 \\ 4 & -3 \end{bmatrix}.$$

$$6. x = 1, y = 3.$$

7. The square is transformed into the line segment  $\overrightarrow{OP}$ , where  $P$  is the point  $(2, 2)$ .

$$8. \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

$$\text{or } x_3 = 7x_0 - y_0 \text{ and } y_3 = x_0 + 7y_0.$$

# CHAPTER FOURTEEN

## VECTORS

A vector quantity is one which requires both a magnitude and a direction in order to specify it completely. A force and a velocity are examples of vector quantities as both a magnitude and a direction are needed to specify them. As force, and velocity, may be represented in magnitude and direction by a straight line, a vector could be defined as a straight line displacement of a point.

Vectors are denoted by small letters **a**, **b**, **c**, etc. in bold type or by a pair of letters with an arrow above them, such as  $\vec{AB}$ ,  $\vec{CD}$ , etc. Whereas a vector **a** is printed in bold type, the scalar quantity *a* is printed in ordinary type. A scalar quantity possesses magnitude only. The magnitude of the vector **a** is denoted by  $|\mathbf{a}|$ , which is called the modulus of **a**. The modulus of a unit vector is unity, and the modulus of a null vector is zero. If two vectors **a** and **b** are equal, their moduli are equal and they act in the same direction. The vector which is equal and opposite to **a** is written as  $-\mathbf{a}$ . The vector equal and opposite to  $\vec{AB}$  is  $\vec{BA}$ .

(a) Consider the parallelogram  $ABCD$ , Fig. 81.

Let  $\vec{AB}$  and  $\vec{BC}$  represent the vectors **a** and **b** respectively.

Then  $\vec{AB} = \vec{DC} = \mathbf{a}$ , and  $\vec{BC} = \vec{AD} = \mathbf{b}$ .

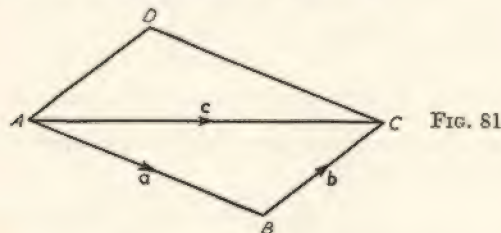


FIG. 81

If  $\vec{AB}$  represents the displacement of a point from *A* to *B*, and  $\vec{BC}$  represents the displacement of a point from *B* to *C*, the

total effect of the two displacements will be the same as the displacement  $\vec{AC}$ .

That is  $\vec{AB} + \vec{BC} = \vec{AC}$ ,

or,  $\mathbf{a} + \mathbf{b} = \mathbf{c}$  . . . . . (1)

This gives the rule for vector addition.

Since, also  $\vec{AD} + \vec{DC} = \vec{AC}$ ,

then,  $\mathbf{b} + \mathbf{a} = \mathbf{c}$  . . . . . (2)

From (1) and (2)  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ .

This means that the commutative law holds for vector addition.

(b) Consider Fig. 82. Let  $\vec{AD}$ ,  $\vec{DC}$ , and  $\vec{CB}$  represent the vectors **a**, **b**, and **c**.

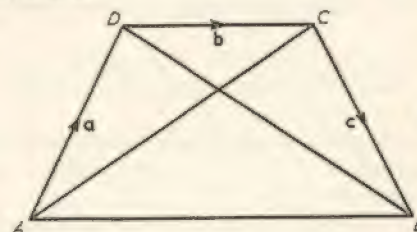


FIG. 82

Let a point be subjected to the successive displacements  $\vec{AD}$  and  $\vec{DB}$ .

These will be equivalent to the displacement  $\vec{AB}$

$$\therefore \vec{AB} = \vec{AD} + \vec{DB}.$$

But displacement  $\vec{DB}$  is equivalent to displacement  $\vec{DC} + \vec{CB}$ .

$$\therefore \vec{AB} = \vec{AD} + (\vec{DC} + \vec{CB}).$$

$$\therefore \vec{AB} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) . . . . . (1)$$

Similarly,

$$\vec{AB} = \vec{AC} + \vec{CB}.$$



But

$$\begin{aligned}\vec{AC} &= \vec{AD} + \vec{DC} \\ \therefore \vec{AB} &= (\vec{AD} + \vec{DC}) + \vec{CB} \\ \therefore \vec{AB} &= (\mathbf{a} + \mathbf{b}) + \mathbf{c} \quad (2)\end{aligned}$$

From (1) and (2)

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}.$$

Therefore the associative law for addition holds for vector addition.

(c) If  $\lambda$  is a scalar quantity, or a number, and  $\mathbf{a}$  is a vector,  $\lambda\mathbf{a}$  is defined as a vector parallel to the vector  $\mathbf{a}$  but of modulus  $\lambda$  times as great.

#### KINDS OF VECTOR

A force needs magnitude and direction to specify it, and also its line of action must be stated. Such a vector is called a *sliding vector*.

The magnitude and direction of magnetic field strength is referred to a particular point. This may be represented by a vector known as a *tied vector*.

There are quantities which are completely specified by magnitude and direction alone, such as the moment of a couple. Such a vector is called a *free vector*.

A system is a vector space if it is an Abelian group with respect to addition, is subject to a scalar multiplication by elements from an associated field of scalars, and if the scalar multiplication obeys the following laws:

1. Distributive law:  $r.(a + b) = r.a + r.b$   
 $(r + s).a = r.a + s.a$
2. Mixed associative law:  $r.(s.a) = (r.s).a$
3. Unity element:  $1.a = a$



FIG. 83

Figure 83 shows the representation of a vector by means of an arrow. The straight line through  $P$  and  $Q$  is called the *line*

of action of the vector.  $P$  is known as the *origin* of the vector, and  $Q$  the *terminus* of the vector. The letter indicating the origin precedes the letter indicating the terminus, and an arrow is placed over the letters. There is a simpler method of denoting a vector. This, as stated earlier in the chapter, consists of a single symbol, a small letter printed in bold type. The letter  $\mathbf{a}$  in bold could denote a vector. The magnitude of a vector is a scalar quantity, and it is never negative. The magni-

tude of the vector  $\vec{PQ}$  is denoted by  $|\vec{PQ}|$ , and the magnitude of the vector  $\mathbf{a}$  is denoted by  $a$  or by  $|\mathbf{a}|$ . Two vectors are equal if they have the same magnitudes and the same directions. If two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are equal, the fact is expressed by  $\mathbf{a} = \mathbf{b}$ .

#### ADDITION OF VECTORS

Vectors obey a certain law of addition, called the law of vector addition, and this is illustrated in Fig. 84.

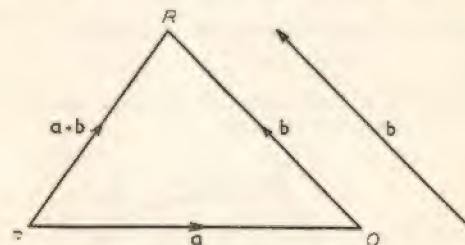


FIG. 84

Let  $\mathbf{a}$  and  $\mathbf{b}$  be two vectors, and let the origin and terminus of  $\mathbf{a}$  be  $P$  and  $Q$  respectively. A vector equal to  $\mathbf{b}$  is now constructed with its origin at  $Q$ . Let its terminus be at the point  $R$ .

Then the sum  $\mathbf{a} + \mathbf{b}$  is the vector  $\vec{PR}$ , and this fact is written  $\mathbf{a} + \mathbf{b} = \vec{PR}$ .

#### THEOREMS ON VECTORS

1. Vectors obey the commutative law of addition.

Therefore,  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ .

*Proof.* Let  $\mathbf{a}$  and  $\mathbf{b}$  be two vectors, as in Fig. 85.

Then,  $\mathbf{a} + \mathbf{b} = \vec{PR} \quad (1)$

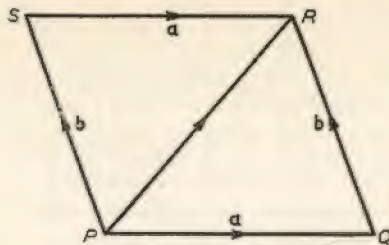


FIG. 85

Now construct a vector equal to  $\mathbf{b}$ , with its origin at  $P$ . Its terminus will fall on  $S$ . Now construct a vector equal to  $\mathbf{a}$  with its origin at  $S$ . The terminus of this vector will fall on  $R$ .

Therefore,  $\mathbf{b} + \mathbf{a} = \overrightarrow{PR}$  . . . . . (2)

From (1) and (2) it follows that,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

2. Vectors obey the associative law of addition.

Therefore,  $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ .

*Proof.* Construct a figure similar to Fig. 86, with vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  as consecutive sides.

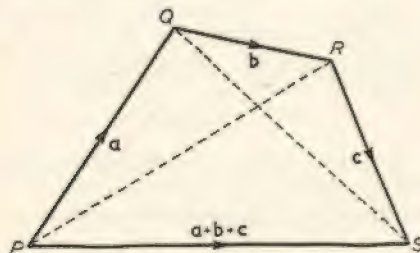


FIG. 86

It will be seen that,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \overrightarrow{PR} + \mathbf{c} = \overrightarrow{PS} \quad . \quad . \quad . \quad (1)$$

It will also be seen that

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = \mathbf{a} + \overrightarrow{QS} = \overrightarrow{PS} \quad . \quad . \quad . \quad (2)$$

Therefore from (1) and (2), the theorem is proved.

# MULTIPLICATION OF A VECTOR BY A SCALAR

By definition, if  $\lambda$  is a positive scalar and  $\mathbf{a}$  is a vector, the quantity  $\lambda\mathbf{a}$  is a vector with magnitude  $\lambda a$  and acting in the same direction as  $\mathbf{a}$ . Also, if  $\lambda$  is a negative scalar,  $\lambda\mathbf{a}$  is a vector with magnitude  $|\lambda| a$ , and acting in the opposite direction to  $\mathbf{a}$ .

3. The multiplication of a vector by a scalar obeys the distributive laws. Therefore,

$$(i) \quad (\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a} \quad . \quad . \quad . \quad (1)$$

$$(ii) \quad \lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b} \quad . \quad . \quad . \quad (2)$$

*Proof* (i) If  $\lambda + \mu$  is positive, both sides of (i) represent a vector of magnitude  $(\lambda + \mu)a$ , acting in the same direction as  $\mathbf{a}$ . If  $\lambda + \mu$  is negative, both sides of (i) represent a vector of magnitude  $|\lambda + \mu| a$ , acting in a direction opposite to  $\mathbf{a}$ .

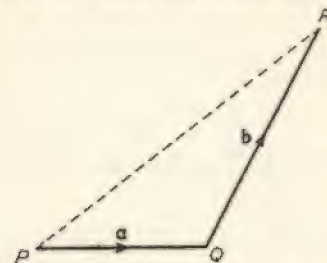


FIG. 87

*Proof* (ii) (a) Let  $\mu$  be positive and let the vectors  $\mathbf{a}$  and  $\mathbf{b}$  be represented as in Fig. 87, and let the vectors  $\mu\mathbf{a}$  and  $\mu\mathbf{b}$  be represented as in Fig. 88.

$$\text{Then} \quad \mu(\mathbf{a} + \mathbf{b}) = \mu\overrightarrow{PR} \quad . \quad . \quad . \quad (3)$$

$$\text{and} \quad \mu\mathbf{a} + \mu\mathbf{b} = \overrightarrow{SU} \quad . \quad . \quad . \quad (4)$$

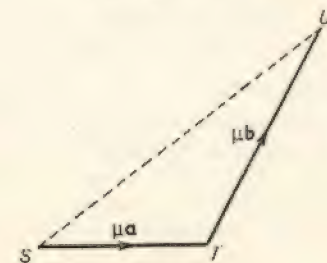


FIG. 88



The two triangles  $PQR$  and  $STU$  are similar, and corresponding sides are proportional, the proportionality constant being  $\mu$ .

Therefore  $\mu \overrightarrow{PR} = \overrightarrow{SU}$ .

Since  $\overrightarrow{PR}$  and  $\overrightarrow{SU}$  have the same directions, and since  $\mu$  is positive, then  $\mu \overrightarrow{PR} = \overrightarrow{SU}$ .

Therefore,  $\mu(a + b) = \mu a + \mu b$ .

(b) Now let  $\mu$  be negative. Then Fig. 88 is replaced by Fig. 89. Again,

$$\mu(a + b) = \mu \overrightarrow{PR},$$

and

$$\mu a + \mu b = \overrightarrow{SU}.$$

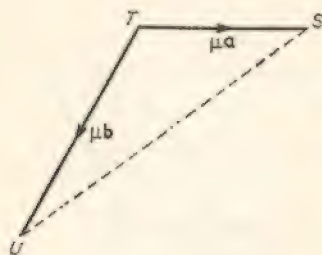


FIG. 89

The triangles  $PQR$  and  $STU$  are still similar, but the proportionality constant is  $|\mu|$ . Therefore  $|\mu| \overrightarrow{PR} = \overrightarrow{SU}$ .

Since  $\overrightarrow{PR}$  and  $\overrightarrow{SU}$  are opposite in direction, and  $\mu$  is negative, then  $\mu \overrightarrow{PR} = \overrightarrow{SU}$ .

Therefore,  $\mu(a + b) = \mu a + \mu b$ .

#### THE SUBTRACTION OF VECTORS

If  $a$  and  $b$  are two vectors, their difference  $a - b$  is defined by the relation  $a - b = a + (-b)$ .

$-b$  is defined as the vector with the same magnitude as  $b$  but acting in the opposite direction.

Figure 90 shows two vectors  $a$  and  $b$ , and their difference  $a - b$ .

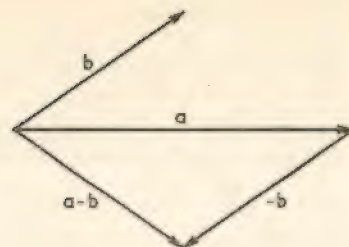


FIG. 90

#### THE SCALAR PRODUCT

Consider two vectors  $a$  and  $b$  with magnitudes  $a$  and  $b$ . Let  $\theta$  be the smallest non-negative angle between  $a$  and  $b$ , as in Fig. 91.

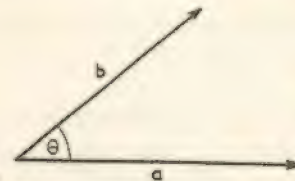


FIG. 91

Then, the scalar product of  $a$  and  $b$  is the scalar  $ab \cos \theta$ .

It is sometimes denoted by  $a \cdot b$ .

Therefore,  $a \cdot b = ab \cos \theta$ .

The scalar product is sometimes called the *dot product*.

#### THEOREMS ON SCALAR PRODUCTS

1. The scalar product is commutative.  
That is  $a \cdot b = b \cdot a$ .
2. The scalar product is distributive.  
That is  $a \cdot (b + c) = a \cdot b + a \cdot c$ .

#### THE VECTOR PRODUCT

Consider two vectors  $a$  and  $b$ , and let the smallest non-negative angle between them be  $\theta$  (Fig. 92).

Then, the vector product of  $a$  and  $b$  is a third vector  $c$  which is defined in terms of  $a$  and  $b$  by the following conditions:

- (a)  $c$  is perpendicular to both  $a$  and  $b$

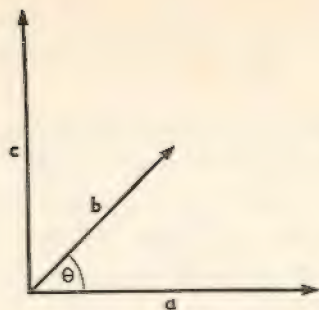


FIG. 92

(b) The direction of  $\mathbf{c}$  is that of the thumb of the right hand when the fingers point in the sense of the rotation of  $\theta$  from the direction of  $\mathbf{a}$  to the direction of  $\mathbf{b}$

(c)  $c = ab \sin \theta$

These conditions will define  $\mathbf{c}$  uniquely. The vector product of  $\mathbf{a}$  and  $\mathbf{b}$  is denoted by  $\mathbf{a} \times \mathbf{b}$ .

Therefore,  $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ .

The vector product is sometimes called the *cross product*.

**Theorem.** The area  $A$  of the parallelogram with vectors  $\mathbf{a}$  and  $\mathbf{b}$  forming adjacent sides is given by  $A = |\mathbf{a} \times \mathbf{b}|$ .

**Proof.** Consider the parallelogram  $OACB$ , Fig. 93, where  $OA$

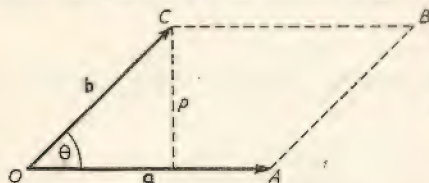


FIG. 93

represents vector  $\mathbf{a}$  and  $OC$  represents vector  $\mathbf{b}$ .  $p$  is the perpendicular from the terminus of  $\mathbf{b}$  on to the line of action of  $\mathbf{a}$ .

Then,  $A = ap$ .

But,  $p = b \sin \theta$ .

Therefore,  $A = ab \sin \theta = |\mathbf{a} \times \mathbf{b}|$ .

**Theorem.** The vector product is distributive.

That is,  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ .

**Theorem.** The vector product is *not* commutative,

because,  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ .

# EXERCISE 13

1. The sides  $\overrightarrow{AB}$ ,  $\overrightarrow{BC}$ ,  $\overrightarrow{CA}$  of a triangle  $ABC$  are denoted by vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ . Prove that  $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ .

2.  $ABC$  is a triangle, and  $D$ ,  $E$  and  $F$  are the mid points of the sides. The medians intersect at  $O$ . Prove that

$$\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

3.  $ABCD$  is a quadrilateral.  $P$  is the mid point of  $BD$ , and  $Q$  is the mid point of  $AC$ . Prove that

$$\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{QP}.$$

4. In the parallelogram  $ABCD$ ,  $M$  and  $N$  lie on the diagonal  $BD$  so that  $BN = MD$ . Show by a vector method that  $ANCM$  is a parallelogram.

5. A ship steams due west at 12 m.p.h. A man walks at 4 m.p.h. relative to the deck from the port to the starboard bow. Find the actual speed and direction of the man's motion.

[12.65 m.p.h.,  $N71^\circ 34'W$ ].

6. Triangle  $ABC$  is right angled at  $B$ , and  $M$  is the mid point of  $AC$ . Taking the position vectors of  $A$  and  $C$  relative to  $B$  as  $\mathbf{a}$  and  $\mathbf{c}$ , express  $\overrightarrow{BM}$  and  $\overrightarrow{AM}$  in terms of  $\mathbf{a}$  and  $\mathbf{c}$ . Hence, show that  $AM = BM$ .

7. Two circles touch externally at  $P$ . A common tangent touches the circles at  $B$  and  $C$ . Prove by a vector method that angle  $BPC$  is a right angle.



# CHAPTER FIFTEEN

## INEQUALITIES

The symbol  $>$  is used to denote 'is greater than', and the symbol  $<$  is used to denote 'is less than'.  $\nless$  means 'is not greater than' and  $\ngtr$  means 'is greater than or equal to'.  $\neq$  means 'is not equal to'. Much care is needed in the use of these symbols, and they cannot be used in the manner used for the sign of equality, except under conditions to be described.

The following statements will be self evident.

- (a) If  $a > b$  and  $b > c$ , then  $a > c$
- (b) If  $a < b$  and  $b < c$ , then  $a < c$
- (c) If  $a > b$ , then  $a + x > b + x$
- (d) If  $a > b$  and  $c > d$ , then  $a + c > b + d$
- (e) If  $a < b$  and  $c < d$ , then  $a + c < b + d$ .

*Cautions.* (i) if  $a > b$  and  $c < d$ , no relationship can be deduced regarding the magnitudes of  $a + c$  and  $b + d$ .

(ii) Never multiply, or divide, inequalities except by numbers known to be positive.

- (f) If  $a > b$ , and if  $\lambda$  is positive, then  $\lambda a > \lambda b$
- (g) If  $a, b, c$  and  $d$  are all positive, and if  $a > b$  and  $c > d$ , then  $ac > bd$ .
- (h) If  $a > b$ , and if  $\lambda$  is negative, then  $\lambda a < \lambda b$ .

Thus multiplication by a negative number reverses the sense of inequality.

Thus, if both sides of an inequality are multiplied, or divided, by the same negative number the inequality sign must be reversed. An inequality may be multiplied, or divided, by a positive number, or have terms transposed from side to side just like an equation.

The modulus of a real number is its numerical value. A modulus is therefore always positive. The 'modulus of  $x$ ' is denoted by the symbol  $|x|$ .

To denote that  $a$  is numerically greater than  $b$ , the modulus notation is used, and it is written  $|a| > |b|$ .

The solution of inequalities.

- (1) Let  $x + 2 > 7$ .
- (1) Subtract 2 from both sides:
- Then  $x > 5$ .

- (2) Let  $2x > -5$ .  
Divide both sides by 2:  
Then  $x > -\frac{5}{2}$ .
- (3) Let  $\frac{1}{5}x < 4$   
Multiply both sides by 5:  
Then  $x < 20$ .
- (4) Let  $-\frac{1}{2}x < -\frac{3}{4}$ .  
Multiply both sides by 2:  
Then  $-x < -\frac{3}{2}$ .  
Multiply both sides by  $-1$ : this changes the sense of the inequality.  
Then  $x > \frac{3}{2}$ .

### PICTORIAL REPRESENTATION OF INEQUALITIES

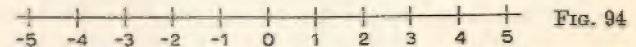


Fig. 94

Figure 94 represents part of the set of natural numbers on a number line. Points on this line to the right are greater than points on the line to the left.

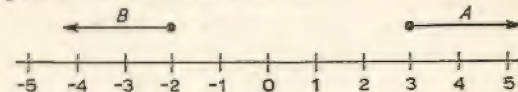


Fig. 95

Figure 95 shows part of the number line representing the natural numbers. The line  $A$  represents the solution set  $x > 3$ , and the line  $B$  represents the solution set  $x < -2$ .

Figure 96 shows a two-dimensional plane figure. The line

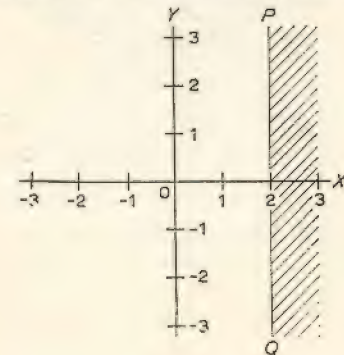


Fig. 96

$PQ$  represents the set of points  $x = 2$ . The shaded region shows the set of points  $x > 2$ . This is an infinite set.

Figure 97 shows the line  $RS$  which represents the set of points  $y = -2$ , and the shaded region shows the set of points  $y < -2$ . This is also an infinite set.

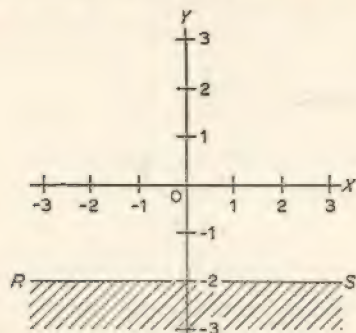


FIG. 97

Figure 98 shows the line  $TU$  which represents the set of points  $y = x$ . The region shaded horizontally shows the set of points,  $y < x$ , and the region shaded vertically shows the set of points  $y > x$ .

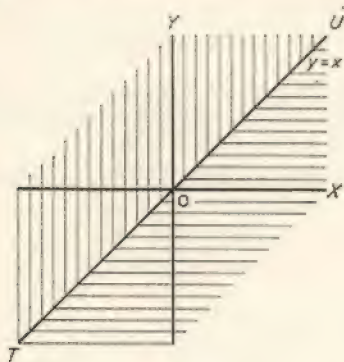


FIG. 98

#### INTERSECTING GRAPHS

In Fig. 99 the line  $AB$  is the graph  $y = x + 2$ . All points on it are members of the solution set of the equation  $y = x + 2$ .

The line  $CD$  represents the graph of  $y = 2x - 3$ . All points on it are members of the solution set of the equation

$$y = 2x - 3.$$

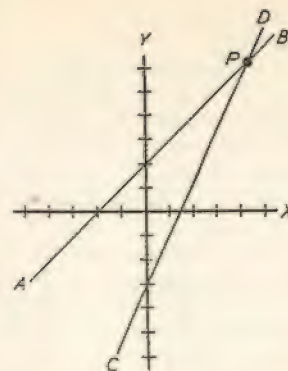


FIG. 99

$P$  is the point of intersection of  $AB$  and  $CD$ . The symbol for the intersection of two sets is  $\cap$ . Therefore the intersection of the solution sets of  $y = x + 2$  and  $y = 2x - 3$  may be represented by,

$$\{(x, y) \mid y = x + 2\} \cap \{(x, y) \mid y = 2x - 3\}.$$

In Fig. 100, the line  $EF$  is the graph of  $y = x + 3$ . All points in the region shaded horizontally are members of the solution set  $y < x + 3$ . This excludes the points actually on  $EF$ .

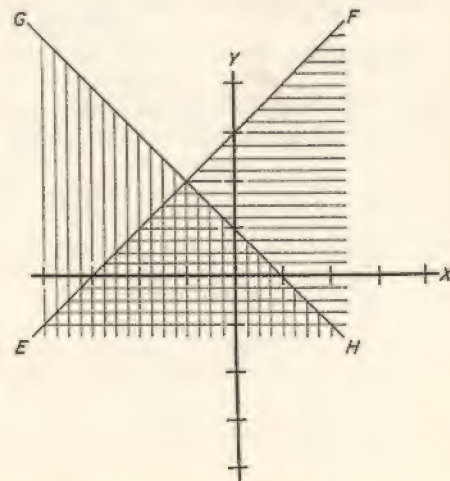


FIG. 100



The line  $GH$  is the graph of  $y = -x + 1$ . All points in the region shaded vertically are members of the solution set  $y < -x + 1$ . This excludes the points actually on  $GH$ .

The intersection of the solution sets  $y < x + 3$ , and  $y < -x + 1$  is the set of points in the doubly shaded region. These may be represented symbolically by,

$$\{(x, y) \mid y < x + 3\} \cap \{(x, y) \mid y < -x + 1\}.$$

#### EXERCISE 14

- Find the values of  $x$  for which  $\frac{5}{2}x + \frac{3}{4} < 2x + \frac{1}{4}$ .
- Find the values of  $x$  for which  $x^2 - 5x + 4$  lies between  $-2$  and  $+2$ .
- Find the values of  $x$  for which  $14x - 20 - 2x^2$  is greater than  $5$ .
- Find the values of  $x$  for which  $2 < \frac{x-7}{x-2} < 3$ .
- Find the ranges of  $x$  for which  $5 - \frac{2}{x-4}$  is greater than zero.
- Find the range of  $x$  for which  $3 + \frac{5}{x+2}$  is less than zero.
- Find the values of  $x$  for which  $-1 < \frac{2x+3}{x-1} < 1$ .

#### ANSWERS TO EXERCISE 14

- For values of  $x$  less than  $-1$ .
- $\frac{1}{2}(5 - \sqrt{17}) < x < 2$ , and  $3 < x < \frac{1}{2}(5 + \sqrt{17})$
- None.
- $-3 < x < -\frac{1}{2}$ .
- $x < 4$  or  $x > 4\frac{2}{5}$ .
- $-2 > x > -3\frac{2}{3}$ .
- $-4 < x < -\frac{2}{3}$ .

#### CHAPTER SIXTEEN

#### FINITE ARITHMETIC

Consider the natural number 3. There are two families of numbers which are not multiples of 3. One family consists of numbers which are 1 more than an integral multiple of 3. For example 7 and 10. The other family consists of numbers which are 2 more than an integral multiple of 3, for example 11 and 17. All integers, therefore, will fall into one of three classes, depending upon whether the remainder, when the integers are divided by 3, is 0 or 1 or 2. These three classes are called residue classes modulo 3.

The three classes may be listed as follows:

- Class 0.  $-12, -9, -6, -3, 0, 3, 6, 9, 12, 15, \dots$
- Class 1.  $-11, -8, -5, -2, 1, 4, 7, 10, 13, 16, \dots$
- Class 2.  $-10, -7, -4, -1, 2, 5, 8, 11, 14, 17, \dots$

The class 0 has properties not possessed by the other classes. Thus, the sum of any two members of the class 0 is also a member of that class. This is the closure property, which means that addition is a binary operation for the class 0. It will also be seen that the class 0 contains the identity element for addition, which is 0.

Furthermore, for every element in the class 0 its negative also belongs to the class.

All this means that the class 0 satisfies all the requirements for being a group. Since addition is commutative it is also an Abelian group.

There is another interesting property of the class 0. If any two members of the class are multiplied, the product is also a member of the class 0. Thus, multiplication is a binary operation for class 0.

As it obeys the distributive law, the class 0 satisfies the requirements for being a ring.

If any member of class 0 is multiplied by any integer, whether in the class or not, the product is a member of class 0. A sub ring which has this property is said to be *ideal*. However, the class 0 does not contain the number 1 which is the unity element for multiplication. Thus a ring need not have a unity element.

### RESIDUE CLASSES MODULO 3

In order to construct a new class, it is necessary to define addition and multiplication.

1. *Definition of Addition.* To add two residue classes, choose one member from each class, and add these members. The class to which the sum belongs is called the sum of the residue classes.
2. *Definition of Multiplication.* To find the product of two residue classes, multiply any member of one class by any member of the other class. The class to which the product belongs is the product of the classes.

It is now possible to construct two tables for class 0, one for addition, and one for multiplication.

Addition Table

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Multiplication Table

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

### SUMMARY OF THE PROPERTIES OF RESIDUE CLASSES MODULO 3

1. A system of residue classes obeys the following five laws.
  - (a) It contains an element called 1.
  - (b) For every member in the system, there is another member, and only one, called its successor.
  - (c) Two distinct members do not have the same successor.
  - (d) There is no member of the system which has 1 as its successor.
  - (e) If a set of elements belonging to the system contains 1, and, for each member that it contains, also contains its successor, then this set contains the whole system.

Therefore, a number system is made up, consisting of only three elements, which is both a group and a ring.

2. It is a group with respect to addition because:

- (a) It contains a zero element, i.e. the class 0,

- (b) For every element in the system, there is also a negative in the system.

The class 0 is a zero element because that class added to any other class leaves it unchanged.

The group is Abelian because the addition operation is commutative.

The system is also a ring because it has a multiplication operation which is distributive with respect to addition.

### RESIDUE CLASSES MODULO 6

When an integer is divided by 6, the remainder is either 0, or 1, or 2, or 3, or 4, or 5. Therefore there are six residue classes modulo 6. The remainder associated with each class is the name of the class. If addition and multiplication are defined as they were in the case of residue classes modulo 3, the following tables may be constructed:

Addition Table

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Multiplication Table

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

### RESIDUE CLASSES MODULO 5

The integers modulo 5 are:

1, 2, 3, 4, 0, 1, 2, 3, 4, 0, 1, 2, 3, ...

These are obtained by dividing the natural numbers by 5 and writing down the remainders.



It will be seen that these additions hold:

$$\begin{array}{ll} 2 + 3 = 0 & 2 + 4 = 1 \\ 3 + 4 = 2 & 4 + 4 = 3 \end{array}$$

The following addition table may be constructed:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

The properties are as follows:

1. *Closure.* The sum of any two elements is also an element.
2. *Identity Element.* An element which does not alter the element upon which it operates is 0.
3. *Inverse Element.* For each element there is one element, and one only, which cancels its effect. Take 0, and add another number, say 3. It is possible to find another number, which is 2, which, on addition, brings the value back to zero; and 2 is the only number which will do this.
4. *Associative Law.* For any three numbers  $a, b, c$ , it is possible to write

$$a + (b + c) = (a + b) + c.$$

Properties 1 to 4 are the properties of a group. So the integers modulo 5 form a group.

It is possible to show that the integers  $\{0, 1, 2, 3, 4\}$  modulo 5 do not form a group under multiplication, but the integers  $\{1, 2, 3, 4\}$  modulo 5 do form a group under multiplication.

#### EXERCISE 15

State whether, or not, the following sets are groups under addition:

1. The integers modulo 8.
2. The integers modulo 6.
3. The integers modulo 3.
4. The real numbers.
5. The negative numbers.

State whether, or not, the following sets are groups under multiplication, in each case exclude 0.

6. The integers modulo 8.
7. The integers modulo 7.
8. The integers modulo 6.

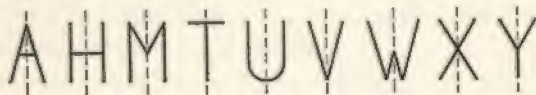
#### ANSWERS TO EXERCISE 15

- |         |         |         |         |
|---------|---------|---------|---------|
| 1. Yes. | 2. Yes. | 3. Yes. | 4. Yes. |
| 5. No.  | 6. No.  | 7. Yes. | 8. No.  |

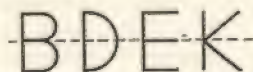
## CHAPTER SEVENTEEN

### SYMMETRY GROUPS

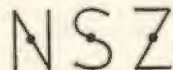
For many objects to be pleasing to the eye, they must possess symmetry and the right proportions. An oblong picture must be such that the length bears a definite ratio to the width for the effect to be most pleasing. The bulge in a vase must be greatest at a certain fraction of the height. Such facts were well known thousands of years ago. Quite primitive people constructed patterns with simple shapes such as circles, triangles, and squares. The Romans used pattern and symmetry in their tessellated floors and in their buildings. Symmetry is closely related to reflection and rotation; it can be obtained by the reflection of a figure in a mirror, so that both sides of the line of symmetry are identical. Some kinds of symmetry are produced by rotation about a point. Many letters of the alphabet possess symmetry.



The letters *A*, *H*, *M*, *T*, *U*, *V*, *W*, *X*, *Y* are obtained by reflection in a vertical mirror.



The letters *B*, *D*, *E*, *K* are obtained by reflection in a horizontal mirror.



The letters *N*, *S*, *Z* are reproduced if they are rotated through an angle of  $180^\circ$ .

#### SYMMETRY OPERATIONS

Let  $p$  denote a half turn about a horizontal axis, or a rotation through  $180^\circ$ , and let  $q$  denote a half turn about a vertical axis.

The operation  $p$ , and the operation  $q$ , on the letter *Z* both produced the letter *N*. Therefore the operation  $p$  followed by the operation  $q$  will reproduce the original *Z*. However, *Z* may be mapped on to itself by a single operation, a rotation of  $180^\circ$ , which is a half turn, in the plane of the paper. Let this latter operation be denoted by the letter  $r$ .

It is now possible to describe the motion symbolically:

$$p[Z] \rightarrow N.$$

Also,

$$q \cdot p[Z] \rightarrow Z.$$

However,

$$r[Z] \rightarrow Z.$$

Therefore,

$$q \cdot p = r.$$

It may be shown, in the same manner, that  $p \cdot q = r$ , and also that  $p \cdot p$ , or  $p^2$ ,  $= I$ .

#### THE IDENTITY ELEMENT AND THE INVERSE ELEMENT

There are letters of the alphabet, such as *F*, *G*, *J*, *L*, *P*, *R*, which are unsymmetrical. These may be brought into line by considering another operation denoted by  $I$ , known as the identity element. Operation with  $I$  leaves the letter unchanged. Since operating by  $p$  twice, or  $p^2$ , restores to the original, it will be seen that  $p^2 = I$ . If an operation  $s$  is reversed by a further operation  $t$ , it is said that the operation  $t$  is the inverse of the operation  $s$ .

#### THE GROUP OF SYMMETRIES OF THE RECTANGLE

- (i) Let  $p$  represent a half turn about a horizontal axis.
- (ii) Let  $q$  represent a half turn about a vertical axis.
- (iii) Let  $r$  represent a half turn in the plane of the paper.

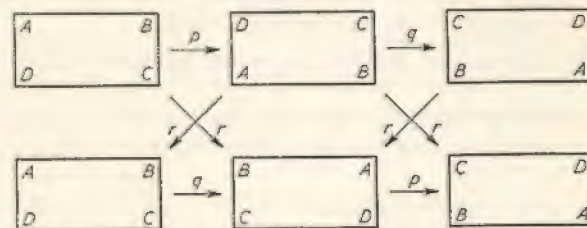


FIG. 101

Figure 101 shows the operations by means of which one position may be changed into another.



If a rectangle is cut from paper and lettered, the following rules may be verified:

$$\begin{aligned} p \cdot q &= q \cdot p = r, & p \cdot r &= r \cdot p = q, \\ q \cdot r &= r \cdot q = p, & p^2 &= q^2 = r^2 = I. \end{aligned}$$

Also, it is evident that,

$$I \cdot p = p; \quad I \cdot q = q; \quad I \cdot r = r; \quad I^2 = I.$$

The results may be placed in a 'multiplication' table

	$\times$	$I$	$p$	$q$	$r$	<i>First operation</i>
	$I$	$I$	$p$	$q$	$r$	
	$p$	$p$	$I$	$r$	$q$	
	$q$	$q$	$r$	$I$	$p$	
	$r$	$r$	$q$	$p$	$I$	

It will be seen that the symmetries of the rectangle have the following properties:

1. The set has a unique identity element  $I$ . This means that there is only one element,  $I$ , which, when multiplied by, or multiplying, any element of the set leaves that element unaltered.

$$\begin{aligned} I \cdot p &= p \cdot I = p; & I \cdot q &= q \cdot I = q; & I \cdot r &= r \cdot I = r; \\ I \cdot I &= I. \end{aligned}$$

2. The set possesses the 'closure' property, since the product of any two elements is also an element of the set.

3. Each element of the set has a unique inverse element. Actually each element is its own inverse element. The effect of the operation  $p$  is cancelled by a second operation  $p$ .

$$\begin{aligned} p \cdot p &= p^2 = I; & q \cdot q &= q^2 = I; & r \cdot r &= r^2 = I; \\ I \cdot I &= I^2 = I. \end{aligned}$$

4. The associative law is obeyed.

Example.  $p \times (q \times r) = p \times p = I.$   
 $(p \times q) \times r = r \times r = I.$   
 $\therefore p \times (q \times r) = (p \times q) \times r.$

and so on for any three elements.

The four properties listed above show that the symmetries of the rectangle form a group.

5. There is an additional property, the commutative law, which is also obeyed, since:

$$p \cdot q = q \cdot p; \quad p \cdot r = r \cdot p; \quad q \cdot r = r \cdot q.$$

Such a group is called a commutative group, or an Abelian group.

#### SYMMETRIES OF THE EQUILATERAL TRIANGLE

This is a good example of a finite group. The symmetries of the equilateral triangle are motions which bring it into coincidence with itself. The best way to study the motions is to begin by cutting out of paper an equilateral triangle, and labelling its vertices  $A$ ,  $B$ , and  $C$ , as in Fig. 102. Mark the letters on both sides of the triangle. Place the triangle as shown, with  $BC$  horizontal and below  $A$ . Each motion which brings the triangle into coincidence with itself will be given a name.

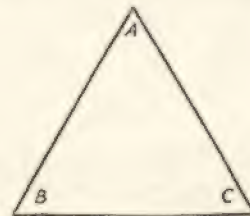


FIG. 102

- One motion will be a clockwise rotation of  $120^\circ$  about the centre of the triangle. This will bring  $B$  to  $A$ ,  $A$  to  $C$ , and  $C$  to  $B$ . Call this motion  $P$ .
- Another motion will be a rotation of  $240^\circ$  in a clockwise direction. This will bring  $C$  to  $A$ ,  $B$  to  $C$ , and  $A$  to  $B$ . Call this motion  $Q$ .
- Another 'motion' will be a rotation of  $0^\circ$ , involving no movement at all. Call this 'motion'  $I$ . There are three further motions which will bring the triangle into coincidence with itself.
- Raise  $BC$  and turn it over the line  $AC$ , which remains where it is, so that the bottom face comes uppermost. See Fig. 103.

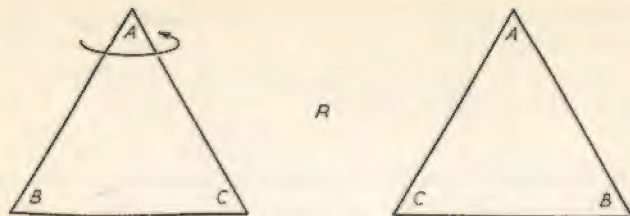


FIG. 103

The motion keeps  $A$  where it is and changes the positions of  $B$  and  $C$ . Call this motion  $R$ .

- (e) Figure 104 shows a similar motion about  $B$ , in which  $A$  and  $C$  change positions. Call this motion  $S$ .

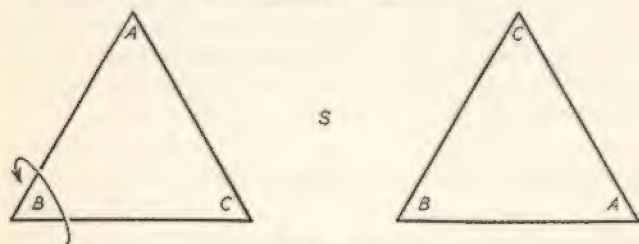


FIG. 104

- (f) Figure 105 shows yet another similar motion, about  $C$ , in which  $A$  and  $B$  change positions. Call this motion  $T$ .

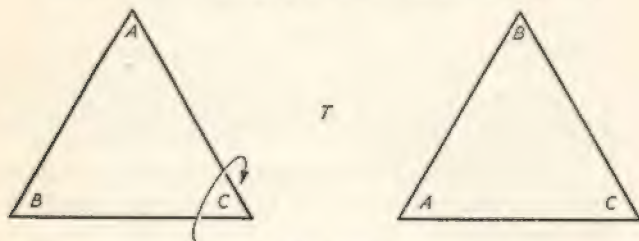


FIG. 105

The foregoing has built up a system of six elements,  $P, Q, I, R, S, T$ . Now define a binary operation  $\oplus$  for the system thus:

If  $X$  and  $Y$  represent any two of these motions, the product

$X \oplus Y$  is the motion which results when the two motions are performed one after the other,  $Y$  being performed first.

The result of performing the operation  $\oplus$  on the equilateral triangle is listed in the following table. The motion which is performed first, and written on the right-hand side of a product, is indicated at the top of the table, and the motion which is performed second, and written on the left-hand side of the product, is indicated down the left-hand side of the table.

$\oplus$	$I$	$P$	$Q$	$R$	$S$	$T$
$I$	$I$	$P$	$Q$	$R$	$S$	$T$
$P$	$P$	$Q$	$I$	$S$	$T$	$R$
$Q$	$Q$	$I$	$P$	$T$	$R$	$S$
$R$	$R$	$T$	$S$	$I$	$Q$	$P$
$S$	$S$	$R$	$T$	$P$	$I$	$Q$
$T$	$T$	$S$	$R$	$Q$	$P$	$I$

In order to show that the symmetries of the triangle form a group it is necessary to show that:

- The operation  $\oplus$  is associative, i.e. that  $X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$ .
- The system has an identity element; this is  $I$ .
- For every element in the system there is an inverse element also in the system. The inverse of  $P$  is  $Q$ , and the inverse of  $Q$  is  $P$ , since  $P \oplus Q = Q \oplus P = I$ .

As these are all verifiable from the table, then the symmetries of the triangle do form a group.

#### ISOMORPHIC GROUPS

Comparing the characteristic groups of symmetries which two different sets of elements possess it will often be observed that there is a similarity of structure. The comparison may be made using the addition and multiplication tables.

When two groups have the same basic structure, they are said to be isomorphic.



# EXERCISE 16

Draw the letter **G** after the following operations have been performed upon it:

1.  $p$
2.  $q$
3.  $r$
4.  $p^2$
5.  $q^2$
6.  $r^2$
7.  $I$
8.  $p \cdot q$
9.  $q \cdot p$

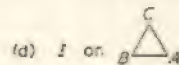
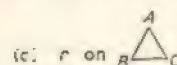
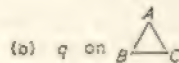
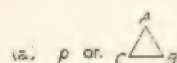
10 The operation  $\omega$  means 'rotate through  $120^\circ$  anticlockwise about the centre of gravity in its own plane'. Perform the operations  $I$ ,  $\omega$ , and  $\omega^2$  on the equilateral triangle



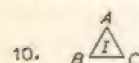
Then complete the multiplication table

$\times$	1	$\omega$	$\omega^2$
1			
$\omega$			
$\omega^2$			

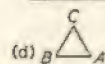
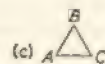
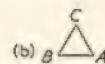
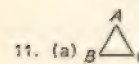
11. Given that  $p$  represents a half turn about the perpendicular from  $A$  to  $BC$ ,  $q$  represents a half turn about the perpendicular from  $B$  to  $AC$ , and  $r$  represents a half turn about the perpendicular from  $C$  to  $AB$ , perform the following operations on the equilateral triangle  $ABC$ .



## ANSWERS TO EXERCISE 16



$\times$	1	$\omega$	$\omega^2$
1	1	$\omega$	$\omega^2$
$\omega$	$\omega$	$\omega^2$	1
$\omega^2$	$\omega^2$	1	$\omega$



## CHAPTER EIGHTEEN SENTENCE LOGIC

### PROPOSITIONS

In this chapter letters are used to represent statements or propositions. Thus, the statement 'I have had breakfast' could be represented by the letter  $a$ , and the proposition 'I have had dinner' could be represented by the letter  $b$ . If it is midday it is likely that  $a$  is a true and  $b$  is a false proposition. In that case we write:

$$a = 1; \quad b = 0.$$

Thus, the symbols 1 and 0 are here used to denote the truth or falsity, respectively, of propositions. Just as in the case of switching algebra, there are no half truths. Thus, a sentence is not a proposition unless it permits of a definite decision of its truth or falsity.

### COMPOUND PROPOSITIONS

The statement 'I have had breakfast and dinner' is a compound proposition containing two simple propositions, both of which are true.

If either meal or both meals were missed, the compound proposition is false.

If the letter  $c$  is used to represent the compound proposition a truth table can be constructed as follows:

$a$	$b$	$c$
0	0	0
0	1	0
1	0	0
1	1	1

This table is the same as the multiplication table for the numbers 1 and 0. It is also the same as the table for a circuit with two switches in series. Thus it is possible to write:

$$ab = c.$$

This combination of propositions is named 'conjunction'. It states 'both  $a$  and  $b$ '. Another combination, called 'disjunction', states 'either  $a$  or  $b$  or both'.

If a man comes home and is asked 'have you had breakfast or dinner', and he replies 'yes', he is stating that he has had one or both of these meals. The compound proposition, say  $d$ , would be false if, and only if, both  $a$  and  $b$  are false. The truth table would then be:

$a$	$b$	$d$
0	0	0
0	1	1
1	0	1
1	1	1

This table is the same as that for a circuit with two switches in parallel. Thus it is possible to write:

$$a + b = d.$$

#### NEGATION OF A PROPOSITION

The negation of  $a$  will be the proposition 'I have not had breakfast', and the symbol used is  $a'$ . It will be obvious that if  $a = 0$ , then  $a' = 1$ , and if  $a = 1$ , then  $a' = 0$ . If a man is asked 'have you had breakfast or dinner', and answers 'no', then he is stating the proposition  $(a + b)'$ . This is the same as the conjunction of the two propositions (i) I have not had breakfast, and (ii) I have not had dinner, which may be written as  $a'b'$ .

Therefore,  $(a + b)' = a'b'$ .

#### IMPLICATION

The symbol  $\Rightarrow$  means 'implies' or 'if . . . then'. Thus  $p \Rightarrow q$  means 'if  $p$ , then  $q$ '.

Suppose we say 'if you get wet you will catch cold'. Here  $p$  = (you get wet) and  $q$  = (you will catch cold). Let the compound expression  $p \Rightarrow q$  be denoted by  $r$ . If, however, one gets wet and does not catch cold, the statement is false, then  $r = 0$ . So two rows of the truth can be written:

$p$	$q$	$r$
1	1	1
1	0	0

If one does not get wet, and does not catch cold, then:

$p$	$q$	$r$
0	0	1

However, one could get a cold without getting wet, so

$p$	$q$	$r$
0	1	1

Thus, the complete truth table will be:

$p$	$q$	$r$
0	0	1
0	1	1
1	0	0
1	1	1

#### STATEMENTS REPRESENTED SYMBOLICALLY

Consider two statements represented by  $p$  and  $q$ .

(i) The negation of  $p$  is written  $\sim p$  (that is, not  $p$ ).

For example, if  $p$  stands for 'the sun is shining', then  $\sim p$  stands for 'the sun is not shining'.

(ii)  $p \wedge q$  represents  $p$  and  $q$ .

For example, if  $p$  stands for 'Jack likes Latin', and  $q$  stands for 'Jack likes French', then  $p \wedge q$  stands for 'Jack likes Latin and French'.

There is an analogy between sets and statements. The laws by which statements are manipulated are the laws of sets. That is why there is a similar form of notation. Thus, if  $P$  is the set of all children who like Latin, and  $Q$  is the set of all children who like French, then  $P \cap Q$  is the set of all children who like Latin and French.

(iii)  $p \vee q$  stands for  $p$  or  $q$  or both.

For example, if  $p$  stands for 'Jack is shouting' and  $q$  stands for 'Jack is clapping', then  $p \vee q$  stands for 'Jack is shouting or clapping or shouting and clapping'.



(iv)  $p \rightarrow q$  means the statement  $p$  implies the statement  $q$ .

For example, if  $p$  stands for 'Smith is a Londoner' and  $q$  stands for 'Smith is an Englishman', then  $p \rightarrow q$ , because the first statement implies the second. However, we cannot write  $q \rightarrow p$ , because not all Englishmen are Londoners.

(v) If we have two statements  $p$  and  $q$  which are such that  $p$  implies  $q$  and  $q$  implies  $p$ , then the statements are said to be equivalent, and this is written  $p \leftrightarrow q$ .

For example, if  $p$  stands for 'triangle XYZ is equilateral', and  $q$  stands for 'triangle XYZ is equiangular', then  $p$  implies  $q$ , and  $q$  implies  $p$ , and we may write  $p \leftrightarrow q$ .

#### TRUTH TABLES

There are occasions when it is not easy to decide whether or not a sentence is true. It is then useful to construct a truth table.

(a) If a statement  $p$  is true ( $T$ ), then the statement  $\sim p$  must be false ( $F$ ), and vice versa. From this a simple truth table can be constructed.

$p$	$\sim p$
$T$	$F$
$F$	$T$

(b) Consider  $p \wedge q$ . There are four possibilities:

- (i)  $p$  and  $q$  are both true
- (ii)  $p$  and  $q$  are both false
- (iii)  $p$  is true and  $q$  is false
- (iv)  $p$  is false and  $q$  is true.

Since ' $p$  and  $q$ ' is true only if  $p$  and  $q$  are both true, the following truth table can be constructed:

$p$	$q$	$p \wedge q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$F$
$F$	$F$	$F$

(c) Consider  $p \vee q$ . In this case either  $p$  or  $q$  or both are true. If  $p$  is false and  $q$  is true, or vice versa, the statement, or disjunction, is true. For example:

(i) 'Either men have two legs or dogs have two legs'. This is a true statement because one of the facts is true. If both statements were false, then the whole sentence would be false.

(ii) 'All men are black or all birds are white' is obviously a false statement. A complication arises when both statements are true. If a man is wearing a green coat and a brown hat, would it be true to say 'he is wearing a green coat or a brown hat'? Now  $p \vee q$  has been defined as  $p$  or  $q$  or both, which is called the inclusive disjunction. It is possible to say 'he is wearing a green coat or a brown hat or both'.

If both statements are true, the compound sentence is true.

If however,  $p$  means 'I am in Australia', and  $q$  means 'I am in Scotland', then the inclusive disjunction is meaningless, and for such cases the symbol  $\vee$  is used, which means  $p$  or  $q$  but not both.

Now it is possible to construct the following truth table:

$p$	$q$	$p \vee q$
$T$	$T$	$T$
$T$	$F$	$T$
$F$	$T$	$T$
$F$	$F$	$F$

(d) The symbol  $\rightarrow$  means 'if . . . then', or 'implies'. So that  $p \rightarrow q$  means 'if  $p$  then  $q$ ' or ' $p$  implies  $q$ '. If  $p$  and  $q$  are both true, then  $p \rightarrow q$  is true. If  $p$  is true and  $q$  is false, then  $p \rightarrow q$  must be false. It is not obvious what happens when  $p$  is false, so we define that whenever  $p$  is false then  $q$  is true, and the following truth table is constructed:

$p$	$q$	$p \rightarrow q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

(e) The symbol  $p \leftrightarrow q$  means 'if  $p$  then  $q$  and if  $q$  then  $p$ '.  $p$  and  $q$  can be equivalent statements only when both are true or when both are false. Thus the following truth table can be constructed:

$p$	$q$	$p \leftrightarrow q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$F$
$F$	$F$	$T$

#### ILLOGICAL REASONING

There is much confusion between a statement and its converse, i.e. between  $p \Rightarrow q$ , and  $q \Rightarrow p$ .

The statement 'If he is ill, then the doctor will call on him', has as its converse: 'If the doctor calls on him, then he is ill'.

If the first statement is accepted as true, it does not necessarily follow that the second statement is true.

The proposition  $p' \Rightarrow q'$  is said to be the inverse of  $p \Rightarrow q$ , but this is not a logical sequence. For example, consider the statement 'If he is an African, he is black'.

The inverse is, 'If he is not an African, he is not black'. The inverse is not true.

However,  $a \Rightarrow b$ , and  $b' \Rightarrow a'$ , have the same truth tables, and so the statements are equivalent.

#### EQUIVALENCE

If one wishes to assert both  $a$  and  $b$ , the symbol  $p \Leftrightarrow q$  is used.  $p \Leftrightarrow q$  means that  $p$  and  $q$  are either both true or both false. Then  $p$  and  $q$  are said to be equivalent. The truth table for  $p \Leftrightarrow q$  is:

$p$	$q$	$p \Leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

*Example.* Suppose  $p$  = triangle  $ABC$  has two equal sides, and  $q$  = triangle  $ABC$  has two equal angles, then it follows  $p \Leftrightarrow q$ .

If one proposition is true, so is the other. If one proposition is false, so is the other.

*Example.* Let  $p$  = triangles  $PQR$  and  $XYZ$  are congruent, and let  $q$  = triangles  $PQR$  and  $XYZ$  are equal in area.

It is known from geometry that  $p \Rightarrow q$ , or, in words, ' $p$  is a sufficient condition for  $q$ '.

This sentence must not be confused with the similar one ' $q$  is a necessary condition for  $p$ '. This sentence means exactly the same as the former. It is another way of describing  $p \Rightarrow q$ . The triangles must be equal in area for them to be congruent. But the equality of area is not sufficient to be sure of congruence, and so ' $q$  is not a sufficient condition for  $p$ '. Also, it is not necessary for two triangles to be congruent for them to be equal in area, and so ' $p$  is not a necessary condition for  $q$ '.

It can happen that a condition is both necessary and sufficient. In order that a triangle be isosceles, it is a necessity that two angles should be equal, and this is a sufficient condition. The symbol  $\Leftrightarrow$  describes the situation. This may also be described by using the phrase 'if and only if'. Then, one says, a triangle is an isosceles triangle if and only if two angles are equal.

#### BOOLEAN ALGEBRA, SETS, AND LOGIC

Comparison of the symbols used in sets, sentence logic and switching circuits.

	Union either-or or both	Intersection and	Complement negation
Sets	$A \cup B$	$A \cap B$	$A'$
Sentence logic	$a \vee b$	$a \wedge b$	$\sim a$
Switching circuits	$a + b$	$a \cdot b$	$a'$

The algebra of sentence logic and the algebra of switching circuits both obey the laws of sets. They are all examples of Boolean algebra.

$U$  is the universal set.  $A$ ,  $B$ , and  $C$  are subsets in Boolean algebra. 1 is written instead of  $\vee$ , and 0 is written instead of  $\emptyset$ .



### Laws of Sets

1.  $A \cap U = A$
2.  $A \cup \emptyset = A$
3.  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4.  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5.  $A \cap (A \cup B) = A$
6.  $A \cup (A \cap B) = A$
7.  $A \cap \emptyset = \emptyset$
8.  $A \cup U = U$
9.  $A \cap A = A$
10.  $A \cup A = A$
11.  $(A \cup B)' = A' \cap B'$
12.  $(A \cap B)' = A' \cup B'$

### Laws of Boolean Algebra

- $a \times 1 = a$
- $a + 0 = a$
- $a(b + c) = ab + ac$
- $a + bc = (a + b)(a + c)$
- $a(a + b) = a$
- $a + ab = a$
- $a \times 0 = 0$
- $a + 1 = 1$
- $a \times a = a$
- $a + a = a$
- $(a + b)' = a' \cdot b'$
- $(ab)' = a' + b'$

The principle of duality is an important one in mathematics, and it is well illustrated by the above laws.

If any rule is taken and  $+$  and  $\times$  are interchanged and 0 and 1 are interchanged, another rule which is true is obtained.

Observation will show that the following rules are dual,

- (i) 1 and 2, (ii) 3 and 4, (iii) 5 and 6, (iv) 7 and 8, (v) 9 and 10, (vi) 11 and 12.

Thus there are six pairs of self dual relationships.

### EXERCISE 17

1.  $p$  represents the statement 'I like sailing' and  $q$  represents the statement 'I am a good swimmer'. Write the meaning of the following:

- (i)  $p \wedge q$       (ii)  $p \vee q$       (iii)  $p \wedge \sim q$   
 (iv)  $\sim p \vee q$       (v)  $\sim p \wedge \sim q$       (vi)  $p \rightarrow q$

2.  $a$  represents the statement 'I am wearing a brown coat' and  $b$  represents the statement 'I am wearing brown shoes'.

Write the meaning of the following:

- (i)  $\sim a$       (ii)  $a \vee \sim b$       (iii)  $b \wedge \sim b$   
 (iv)  $a \vee \sim b$       (v)  $a \wedge \sim b$       (vi)  $a \rightarrow b$

### ANSWERS TO EXERCISE 17

1. (i) I like sailing and I am a good swimmer.  
 (ii) I like sailing or I am a good swimmer (or both).

- (iii) I like sailing and I am not a good swimmer.  
 (iv) I do not like sailing or I am a good swimmer (or both).  
 (v) I do not like sailing and I am not a good swimmer.  
 (vi) I like sailing implies I am a good swimmer.
2. (i) I am not wearing a brown coat.  
 (ii) I am wearing a brown coat or I am not wearing brown shoes.  
 (iii) I am wearing brown shoes or I am not wearing brown shoes.  
 (iv) I am wearing a brown coat or I am not wearing brown shoes.  
 (v) I am wearing a brown coat and I am not wearing brown shoes.  
 (vi) I am wearing a brown coat implies I am wearing brown shoes.

## POINTS, RELATIONS AND FUNCTIONS

The set of values of  $x$  for which  $x$  is greater than three is written symbolically as  $\{x \mid x > 3\}$ . If the set  $\mathcal{E}$  is  $\{1, 2, 3, 4, 5, 6\}$ , then for  $x \in \mathcal{E}$ ,  $\{x \mid x > 3\}$  is the set  $\{4, 5, 6\}$ . If  $x$  may take values which are positive whole numbers between 0 and 10, that is  $x \in \mathcal{E}$ , where  $\mathcal{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ , then the set  $\{x \mid x > 3\}$  is the set  $\{4, 5, 6, 7, 8, 9\}$ . If however  $\mathcal{E}$  is the set  $\{0, 1, 2\}$ , then  $\{x \mid x > 3\} = \emptyset$ , which is the null set.

*Example.* Assuming  $x$  may take only positive integral values, list the set  $\{x \mid x > 2\} \cup \{x \mid x < 3\}$ .

$$\begin{aligned}\{x \mid x > 2\} \cup \{x \mid x < 3\} &= \{3, 4, 5, 6, \dots\} \cup \{1, 2\} \\ &= \{1, 2, 3, 4, 5, 6, \dots\}.\end{aligned}$$

## ORDERED PAIRS

Let  $x$  and  $y$  be typical elements of two sets  $X$  and  $Y$ , so that  $x \in X$ , and  $y \in Y$ . The set  $(x, y)$  is an ordered pair, and is so called because of the importance of the order in which the elements are written. For example, the points  $(3, 4)$  and  $(4, 3)$  are not the same. This is because, by convention, the  $x$ -coordinate of a point is placed before its  $y$ -coordinate.

## RELATIONS

A set of ordered pairs is called a relation, because there is usually a definite connection between the elements of each pair and applying to all pairs. To draw the graph of a relation, two scales are needed, one for the first elements and one for the second. For this purpose two lines are drawn at right angles, and they are called axes. The set of first elements is marked along the horizontal axis, and the set of second elements is marked along the vertical axis. The ordered pairs are then represented by points on the diagram.

The domain of a relation is defined as the set of all the first elements in the relation, while the set of all the second elements in the relation is called the range of that relation.

If  $p$  represents any first element of a relation, and  $q$  represents a corresponding second element, then the ordered pair  $(p, q)$  represents a typical member of the relation. The relation mentioned here is a binary relation because it relates two things.

It is possible however to have relations connecting more than two things.

A first element may have many second elements associated with it. It is then called a many-to-one relation. A second element can have only one first element associated with it.

## LINES AND REGIONS

Consider the set of ordered pairs  $\{(x, y) \mid x + y = 3\}$  in which there is no restriction on the values of  $x$  and  $y$ . The set will contain an infinite number of members, each of which will lie on the straight line  $AB$  (Fig. 106), which is called the graph of the equation  $x + y = 3$ .

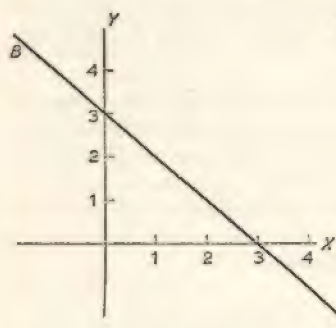


FIG. 106

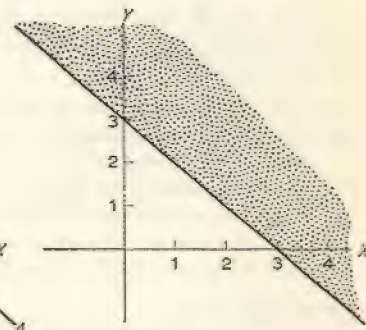


FIG. 107

If  $x$  and  $y$  are real numbers, the set of ordered pairs  $\{(x, y) \mid x + y > 3\}$  will fill the whole plane shown shaded in Fig. 107. This region will not contain the boundary line  $x + y = 3$ . The boundary will only be included if the set were  $\{(x, y) \mid x + y \geq 3\}$ .

## FUNCTIONS

A relation in which no two different ordered pairs have the same first member is known as a function. Thus  $\{1, 1\}, (2, 3)\}$  is a function, but not so  $\{(1, 2), (1, 3), (2, 3)\}$ . A function  $f: A \rightarrow B$ , from a set  $A$  into a set  $B$ , is a correspondence that assigns to each element  $a \in A$  a unique element in  $B$  denoted by  $f(a)$ . The symbol  $f(a)$  is called the value of  $f$  at  $a$ . The range of a function  $f: A \rightarrow B$  is a subset of the co-domain  $B$  comprising elements that actually correspond to some element of  $A$ .



*Example.*  $\{(x, y) \mid y = x + 8\}$  is a function because if a list of ordered pairs is drawn up, there are no two pairs which have the same first but different second element. To each value of  $y$  there is one and only one value of  $x$ . In this particular case, there is a correspondence between the set of real number values of  $x$  and the set of real values of  $y$ .

*Example.*  $\{(x, y) \mid y > x + 2\}$  is not a function because if a list were made of ordered pairs, it would contain such as  $(1, 3)$  and  $(1, 4)$ , etc.

*Example.*  $\{(x, y) \mid x = 10\}$  is not a function because a list of ordered pairs would contain such as  $(10, 1)$ ,  $(10, 2)$ , etc.

*Example.*  $\{(x, y) \mid y = 7\}$  is a function, because no different ordered pairs of the set  $(1, 7)$ ,  $(2, 7)$ ,  $(3, 7)$ ,  $(4, 7)$  . . . contain the same first element.

*Example.*  $\{(x, y) \mid y^2 = x\}$  is not a function because a list of ordered pairs would contain such as  $(4, -2)$  and  $(4, +2)$ . However, if the parts of the curve  $y^2 = x$  in the two quadrants are treated separately so that we have

$$(i) \{(x, y) \mid y^2 = x, y > 0\}$$

$$\text{or } (ii) \{(x, y) \mid y = +\sqrt{x}\},$$

then each of (i) and (ii) forms a function.

#### CORRESPONDENCE

Correspondence between sets has considerable importance in mathematics. That type of correspondence in which every element in a domain has exactly one partner in the co-domain is called a function.

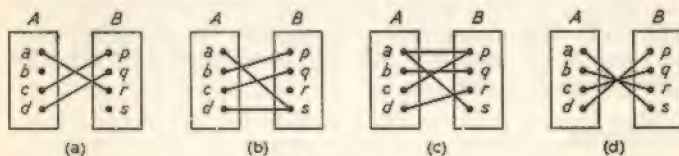


FIG. 108

Consider the correspondences in Fig. 108. In Fig. 108(a) the element  $b$  of  $A$  has no corresponding element in  $B$ . Therefore the matching is not a function from  $A$  to  $B$ .

In Fig. 108(b) each element of  $A$  does have a matching value in  $B$ . Also, no element of  $A$  has more than one corresponding value in  $B$ . Therefore the matching is a function from  $A$  to  $B$ .

In Fig. 108(c) every element in  $A$  has a corresponding element in  $B$ , but to the element  $a$  in  $A$  both  $p$  and  $s$  are matched. Therefore the correspondence is not a function from  $A$  to  $B$ .

In Fig. 108(d) every element of  $A$  has exactly one corresponding element in  $B$ , no more and no less. Therefore the correspondence is a function from  $A$  to  $B$ .

In the above, the set  $A$  is called the domain of the function, and the set  $B$  is called the co-domain. A function is a pairing of the elements of two sets in which every element of the domain is used exactly once. Every element of the domain must be used, and used only once.

#### INVERSE CORRESPONDENCE

If there is a function from  $A$  to  $B$ , there is a correspondence or matching from  $A$  to  $B$ . If there is a correspondence from  $B$  to  $A$ , that is in the opposite direction, it is known as the inverse correspondence. Considering the examples in Fig. 108, it will be seen that the inverse correspondence is not necessarily a function. The symbol for the inverse correspondence is  $f^{-1}$ .

$f(a)$  is sometimes called the image of  $a$  under  $f$ , and  $f^{-1}(b)$  is called the inverse image, when  $b$  is the corresponding image in the co-domain.

#### MAPPING

A mapping is a function. Just as the points or elements on a map correspond to points or elements of a piece of land. Every field or every building will be represented by one and only one part of the map. Let the elements corresponding to buildings be denoted by  $B$ , and let corresponding parts of the map be denoted by  $M$ . Then, the mapping can be symbolized by

$$f: B \rightarrow M.$$

If  $b \in B$  and  $m \in M$ , then it is possible to write

$$f(b) = m.$$

In this particular mapping there is one:one correspondence. Also,  $m$  is the image of  $b$  in the set  $M$ .

#### EXERCISE 18

State whether the following sets of ordered pairs are relations or functions:

- $\{(x, y) \mid x + y = 5\}$
- $\{(x, y) \mid x + y > 5\}$
- $\{(x, y) \mid y - 5 = 0\}$
- $\{(x, y) \mid x + 3 = 0\}$



5.  $\{(x, y) \mid y - x^2 = 0\}$       6.  $\{(x, y) \mid xy = 9\}$   
 7.  $\{(x, y) \mid x^2 + y^2 = 36\}$       8.  $\{(x, y) \mid x^2 + y^2 < 36\}$

In each of the following functions, state whether or not the mapping is 1:1 and state the domain and range:

9.  $\{(x, y) \mid x + y = 6\}$       10.  $\{(x, y) \mid y - 6 = 0\}$   
 11.  $\{(x, y) \mid y^2 - x = 0\}$       12.  $\{(x, y) \mid y = x^2\}$   
 13.  $\{(x, y) \mid xy = 9\}$       14.  $\{(x, y) \mid y = x^2 + 1\}$

#### ANSWERS TO EXERCISE 18

1. A function      2. Not a function      3. A function  
 4. Not a function      5. A function      6. A function  
 7. Not a function      8. Not a function

In the following,  $z$  is the set of real numbers and  $z^+$  is the set of positive real numbers;

9. 1:1;  $z$ ;  $z$       10. Not 1:1;  $z$ ; 4  
 11. 1:1;  $z$ ;  $z$       12. Not 1:1;  $z$ ;  $z^+$   
 13. 1:1;      14. Not 1:1;  $z$ ; all real numbers  $> 1$ .

## CHAPTER TWENTY

### STATISTICS

Statistics is the study of the collection and meaning of data. Number facts are called data. The complicated modern way of life often involves statistics; used by scientists to solve problems, by politicians to 'prove' arguments, and by commerce to obtain information about business. They deal with sport, the weather, prices, wages, and numerous other things. They are used to influence our thoughts and our lives. The arguments that smoking causes lung cancer are entirely statistical. The number of people who vote for a particular party, or use a particular brand of article, or watch a particular television programme, is deduced for the whole country by taking a sample; by putting questions to a certain number of people, and then using proportion. It is important to use discretion in 'sampling' because 'bias' may be introduced. Statistics are often misinterpreted deliberately, to produce a desired result. Sometimes they are misinterpreted because certain factors are not taken into account. All statistics must be viewed with great care and the true meaning can only be obtained by asking such questions as: who collected the data, what was the source of the data, what was the method of collecting the data, and so on. School reports or marks obtained in an examination, can be particularly misleading. A student who obtains 60% in mathematics could be top of the class or bottom of the class. The top boy in form 4C may be weaker than the bottom boy in form 4A. Yet the former boy may have 'good' on his report, and the latter may have 'weak' on his report. The marks obtained in an examination taken by many thousands may be most misleading. A 'pass' could range between 50% to 100%, and a 'fail' be anything from 0% to 49%. Some subjects are easier to pass than others. Also we have 'off' days. One can get a reasonably true picture of the performance of a fit pupil only by considering how many pupils obtained each mark. The number of times each mark appears is called the frequency, and this kind of summary is known as a frequency distribution.

#### PICTORIAL REPRESENTATION OF DATA

Most people find it easier to gain information visually from pictures rather than from the written word, although pictures



can be drawn in such a way as to misrepresent the facts. Graphs, however, are most useful in statistics, and there are several kinds which may be used: (a) circle graphs, or pie charts, (b) line graphs, (c) bar graphs, (d) rectangular distribution graphs, (e) dot frequency graphs, (f) histograms.

1. *Pie Charts.* Figure 109 shows a pie chart giving in the various sectors the proportional amounts spent by a family from the whole income.



FIG. 109

2. *Bar Graphs.* Figure 110 shows the same information given in the form of a bar graph. The rectangles have the same width, but the heights of the rectangles are proportional to the amounts spent.

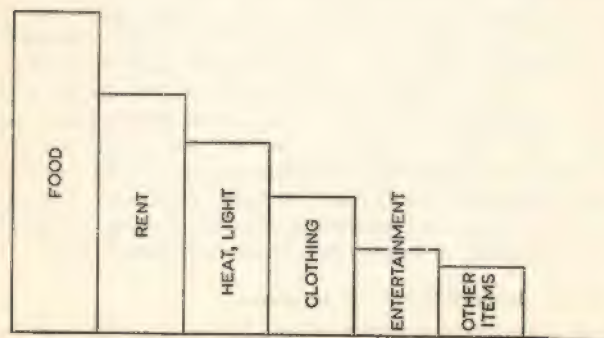


FIG. 110

3. *Histograms.* The histogram looks at first sight like an *upright* bar chart; but it is not used in the same circumstances, and it is constructed on a different principle. It is used when the data consists of groups of numerical measurements and when a frequency can be associated with any numerical range of the measurements covered by the data. In a bar chart the length of the bars represents the frequencies, while in a histogram the area of the columns represents the frequencies. The vertical scale of the histogram is so constructed that the areas (representing frequencies) can be calculated. The vertical scale represents the given frequencies divided by the numerical range of the corresponding data.

This is illustrated by the following example:

*Example.* The following table gives the salary ranges in dollars to be found in a certain factory, and the number of men whose earnings fall into each group. Plot the figures on a histogram as shown in Fig. 111.

\$	0-1500	1501-2000	2001-2500	2501-3000	3001-3500
Men	8	20	15	5	3

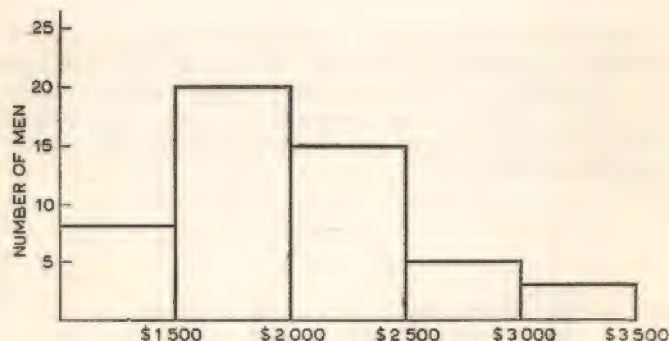


FIG. 111

4. *Frequency Polygon.* Suppose a man goes fishing and during a day catches 14 fish. Suppose the weights of the fish to be, in lb; 4,  $3\frac{1}{2}$ , 2,  $3\frac{1}{2}$ ,  $4\frac{1}{2}$ , 5,  $3\frac{1}{2}$ , 3,  $2\frac{1}{2}$ ,  $4\frac{1}{2}$ , 4, 3,  $3\frac{1}{2}$ , 3. These weights

may be tabulated as follows, according to the frequency of occurrence:

<i>lb</i>	2	2½	3	3½	4	4½	5
<i>Frequency</i>	1	1	3	4	2	2	1

The results may be represented on a frequency polygon as shown in Fig. 112. The actual points plotted are joined by straight lines.

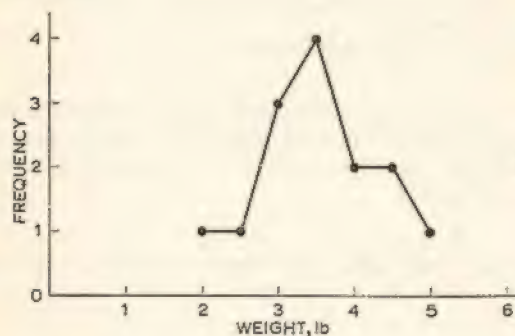


Fig. 112

5. *Normal Distribution Curve.* If the marks obtained in an examination are plotted against their frequency, just as is done for the construction of a frequency distribution graph, a regular curve will be obtained, provided that the setting and marking of the question papers and scripts are satisfactory. The curve is illustrated in Fig. 113.

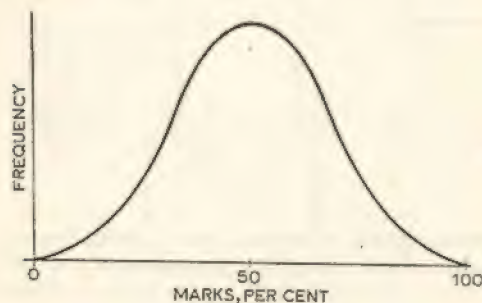


Fig. 113

It is bell shaped and the peak should come at 50%, which is the mark one would expect of the average child. Actually, if such a curve is plotted, it gives some indication of the standard of the paper and of the marking. If the peak were to the left of 50%, it would mean that either the questions were too difficult or the marking was too severe. The distribution would then be said to be skew.

The same shape of graph would be obtained if the weights, or heights, of a number of people were plotted.

#### MEAN VALUE

The mean value of a set of numbers or quantities is usually understood to be the arithmetic mean.

(1) If there are  $n$  numbers  $x_1, x_2, x_3, x_4, \dots, x_n$ , then the arithmetic mean is

$$\frac{x_1 + x_2 + x_3 + x_4 + \dots + x_n}{n} = \frac{\sum_{r=1}^n x_r}{n}$$

(2) If the frequencies of the observations  $x_1, x_2, x_3, \dots, x_n$  are  $f_1, f_2, f_3, \dots, f_n$ , then the mean value, or arithmetic mean is

$$\frac{f_1 x_1 + f_2 x_2 + f_3 x_3 + \dots + f_n x_n}{f_1 + f_2 + f_3 + \dots + f_n} = \frac{\sum f_r x_r}{\sum f_r}$$

#### THE MODE

The mode may be regarded as the most popular value, or the most common member of a set of numbers or quantities. Suppose that a tobacconist stocked 5 different brands of cigarette. He keeps a record of the number of each brand sold, and the highest number of any particular brand sold is called the mode. For simplicity, consider 5 brands,  $A, B, C, D, E$ , and suppose that the number of each brand sold, is  $A, 18; B, 25; C, 36; D, 29; E, 21$ . Then the mode is 36.

#### MEDIAN

The median value is the value of the middle element when the elements are arranged in order of magnitude.

Let an examination candidate gain the following marks in 8 subjects: 45, 47, 51, 56, 58, 60, 63, 69.

The marks have been arranged in order of magnitude, and the median is between 56 and 58, which is 57. Should it happen that there were an odd number of marks, add one and then divide by two in order to find the position of the middle rating.



## RELATION BETWEEN MEAN, MEDIAN AND MODE

For a normal distribution the mean, median, and mode coincide, but for a skew distribution they are distinct values.

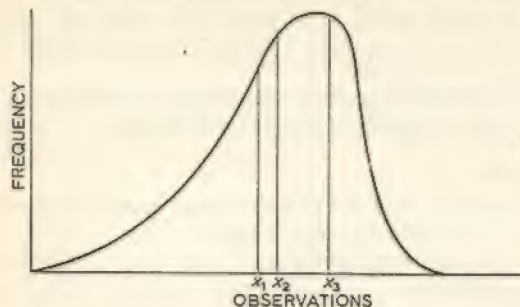


Fig. 114

The positions of the values are shown in the skew distribution, Fig. 114.

(i) The mean value is  $x_1$ , and is the  $x$ -coordinate of the centroid of the area under the curve.

(ii) The median value is  $x_2$ , and corresponds to the ordinate which bisects the area under the curve.

(iii) The mode is  $x_3$ , and is the  $x$ -coordinate corresponding to the maximum value of the frequency.

The relationship between the three quantities is approximately as follows:

$$\text{mode} - \text{median} = 2(\text{median} - \text{mean}).$$

$$\text{i.e.} \quad x_3 - x_2 = 2(x_2 - x_1).$$

$$\text{i.e.} \quad x_3 + 2x_1 = 3x_2.$$

The arithmetic mean of several quantities may not be a typical quantity if the data include a few extreme values in one direction. The mode is sometimes thought to be the most typical score of all because it is the most frequently occurring, but it does not take into account the other values in the data. It is easy to find although there may be several other values in the set of data which satisfy the definition of a mode. The median is the middle score and it is not influenced by the other values in the data except in so far as they are either above or below the median. If a score is below the median, it is immaterial whether just below or a lot below. If the scores are concentrated in distinct and widely separated groups, the median could have little value as a measure of central tendency.

## PERCENTILE RANK

A measurement usually only has a meaning when it is compared with a group of similar measurements. One way of showing comparison is to give the rank of the score from the top. Suppose one says that Johnny is eighth in his class. This has no meaning unless it is known how many are in Johnny's class. If there were only eight pupils in the class, he would be bottom, but if there were 250, he would have a high rank. There would be 242 below him. If a fraction is made it will be  $\frac{242}{250}$ ; changing it to a percentage, 96.8 is obtained. It is said that Johnny's percentile rank is 96.8.

## SCATTERING, RANGE, DEVIATION

Measures of central tendency such as the mean, mode, median, of a set of data give a single number which does not always give a complete picture of the data. The average rainfall of a country forms a part of the picture of the climate, but it does not tell us whether the whole rainfall occurred in three or four very heavy downpours or whether it was fairly evenly spread over several months. A more complete picture is presented if the range is stated. The range of a set of data is the difference between the largest and smallest scores of the set. This gives a simple measure of dispersion. However, the range also has shortcomings as a measure of dispersion, and so a measure of dispersion which is related to the mean is developed as follows: First, list each score of a set of data, and let the score be denoted by  $X$ . Let the calculated mean be  $M$ . Now find out how each score differs from the mean, i.e.  $X - M$ . Square this and obtain  $(X - M)^2$ . The squaring takes care of the possibility of the occurrence of negative values of  $X - M$ . Now take the arithmetic mean of the squares of the deviations by dividing the sum,  $\Sigma(X - M)^2$ , of the squares by the number of scores,  $N$ .

This gives  $\frac{\Sigma(X - M)^2}{N}$ . Now find the square root of the result,

$\sqrt{\frac{\Sigma(X - M)^2}{N}}$ . This will represent a measure of the deviation of the scores from the mean. It is called the standard deviation, denoted by  $\sigma$ .

Therefore,

$$\sigma = \sqrt{\frac{\Sigma(X - M)^2}{N}}.$$

# EXERCISE 19

- Find the mean and median of: 7, 6, 6, 5, 7, 9, 9, 8, 6, 5, 4, 8, 7, 6, 3, 4, 5, 5, 4, 3.
- Find the mean and median of: 7.7, 6.5, 4.6, 5.7, 7.4, 9.3, 6.2, 8.5, 10.7.
- Find the mean, median, and modal number of children per family from the following table:

<i>Families</i>	15	30	25	19	8	2	1
<i>Children per family</i>	0	1	2	3	4	5	6

- Find the mean, median, and modal number of persons per house from the following table:

<i>Houses</i>	26	113	120	95	60	42	21	14	5	4
<i>Persons per house</i>	1	2	3	4	5	6	7	8	9	10

## ANSWERS TO EXERCISE 19

- 5.85; 6
- 7.4; 7.4
- 1.85; 2; 1
- 3.78; 3; 3

# CHAPTER TWENTY-ONE

## PROBABILITY

The word probability sounds somewhat vague, but the subject is as exact as any other branch of mathematics. It has nothing to do with such uncertainties as predicting the weather, but deals with such problems as the chance of, say, picking an ace from a pack of cards. There is a great deal of betting and gambling nowadays: people's attitude is that 'someone must win'. But the mathematical chances of winning, perhaps a football pool, are negligible. Mathematicians have decided upon the following scale for measuring probability: 1 means that the occurrence is bound to happen; 0 means that the occurrence cannot happen. Probabilities are written in decimal or fractional form. A probability of one in ten, therefore, could be written as 0.1 or  $\frac{1}{10}$ . So the probability of tossing a coin to come down heads, or tails, could be 0.5 or  $\frac{1}{2}$ ; the probability of picking diamonds from a pack of cards is 0.25 or  $\frac{1}{4}$ .

*Example 1.* Find the probability of getting (i) two heads, (ii) a head and a tail, with two throws of a coin. There are the following possibilities:

<i>First throw</i>	<i>Second throw</i>
H	H
H	T
T	H
T	T

(i) Thus there is 1 possibility out of 4 that two heads are thrown. So the probability is  $\frac{1}{4}$  or 0.25.

(ii) There are 2 possibilities in 4 that a head and a tail are thrown. So the probability is  $\frac{2}{4}$  or  $\frac{1}{2}$  or 0.5.

*Example 2.* Find the probability of throwing (i) three heads, (ii) two heads and one tail, (iii) three heads or three tails, with the tossing of three coins. There are the following possibilities:

<i>First throw</i>	<i>Second throw</i>	<i>Third throw</i>
H	H	H
H	T	H
H	H	T
H	T	T
T	H	H
T	T	H
T	H	T
T	T	T



It will be seen that

- (i) the probability of throwing three heads is  $\frac{1}{8}$ ,
- (ii) the probability of throwing two heads and a tail is  $\frac{3}{8}$ ,
- (iii) the probability of throwing three faces alike is  $\frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$ .

#### THE ADDITION LAW

This law states that if the probabilities of  $n$  naturally exclusive events happening are  $p_1, p_2, p_3, p_4, \dots, p_n$ , then the probability that one of the events will occur is

$$p = p_1 + p_2 + p_3 + p_4 + \dots + p_n$$

*Example 1.* Find the probability of throwing a 5 or a 6 with an ordinary die (singular of dice).

The probability of tossing a 5 is  $\frac{1}{6}$ .

The probability of tossing a 6 is  $\frac{1}{6}$ .

Therefore the probability of tossing a 5 or a 6 is  $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ .

*Example 2.* Find the probability of tossing not more than 4 with one toss of a die.

The probability of throwing 1 is  $\frac{1}{6}$

The probability of throwing 2 is  $\frac{1}{6}$

The probability of throwing 3 is  $\frac{1}{6}$

The probability of throwing 4 is  $\frac{1}{6}$ .

Therefore the probability of throwing either 1 or 2 or 3 or 4 is  $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$ .

*Example 3.* Let a box contain 6 black balls and 4 white balls. Find the probability of taking out a black or a white ball.

The probability of taking out a black ball is  $\frac{6}{10}$ .

The probability of taking out a white ball is  $\frac{4}{10}$ .

Therefore the probability of taking out a black or a white ball is  $\frac{6}{10} + \frac{4}{10} = \frac{10}{10} = 1$ . This is the expected result, because the box contains only white and black balls.

#### THE MULTIPLICATION LAW

This law states that if the probabilities of  $n$  independent events are  $p_1, p_2, p_3, p_4, \dots, p_n$ , then the probability of all the events occurring is

$$p = p_1 \times p_2 \times p_3 \times p_4 \times \dots \times p_n$$

Events are independent when no event can influence any future events.

*Example 1.* Find the probability that 4 heads will be thrown in 5 throws of a coin.

The probability of throwing a head on the 1st throw =  $\frac{1}{2}$

The probability of throwing a head on the 2nd throw =  $\frac{1}{2}$

The probability of throwing a head on the 3rd throw =  $\frac{1}{2}$

The probability of throwing a head on the 4th throw =  $\frac{1}{2}$

The probability of throwing a head on the 5th throw =  $\frac{1}{2}$ .

Therefore the possibility of throwing five heads in a row is  $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$ .

*Example 2.* Find the probability of throwing two 3's with two throws of a dice.

The probability of throwing 3 on the 1st throw =  $\frac{1}{6}$

The probability of throwing 3 on the 2nd throw =  $\frac{1}{6}$ .

Therefore, the probability of throwing two 3's is  $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ .

*Example 3.* A die is thrown twice. Find the probability that the first throw is less than, or equal to, 4 and the second throw is less than, or equal to, 3.

The probability of the first throw being less than, or equal to, 4 is  $\frac{4}{6}$ .

The probability of the second throw being less than, or equal to, 3 is  $\frac{3}{6}$ .

Therefore, the probability of both events happening is

$$\frac{4}{6} \times \frac{3}{6} = \frac{12}{36} = \frac{1}{3}$$

#### THE PROBABILITY TREE

This is a method of calculating probabilities when there is an equal chance of a sample being rejected or accepted. The probabilities of the event's happening are indicated on the various branches; the final probability of its happening is obtained by multiplying the final probability in the branch.

The method can be used only when one single event can occur at a time, that is, when the events are mutually exclusive. Consider the tossing of a coin,  $H$  means that a head appears, and  $T$  means that a tail appears. Figure 115 shows the tree obtained by the tossing of a coin.

The tree shows that the probability of

(i) three tails occurring is  $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

(ii) a tail and 2 heads occurring is

$$(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) = \frac{3}{8}$$

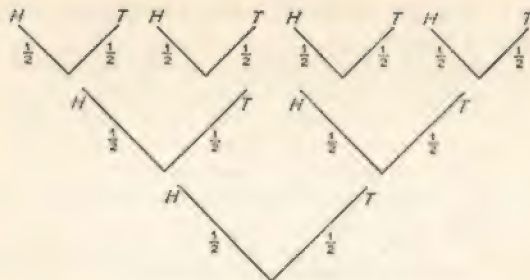


FIG. 115

### EXERCISE 20

1. Two dollars and four other coins are put in line at random. Find the probability that the end coins are both dollars.
2. Nine balls are drawn at random from a bag containing 11 white and 9 black balls. Find the probability that 5 are white.
3. Ten cents and two dollars are placed in a circle at random. Find the odds against the two dollars being placed together.
4. Find the probability of getting a double six at least once in ten throws of two dice.
5. A bag contains 8 white balls and 6 black balls. Five balls are drawn at random. Find the probability that (i) three balls are white, (ii) three or more balls are white.

### ANSWERS TO EXERCISE 20

1.  $\frac{1}{15}$
2.  $\frac{58,212}{167,960} \approx 0.347$
3. 9:2
4. 0.245
5. (i)  $\frac{840}{2002}$ , (ii)  $\frac{1316}{2002}$

### LIST OF SYMBOLS

$\{ \}$	the set of
$\in$	is a member of; belongs to
$\notin$	does not belong to
$\exists$	for some numbers
$\mathcal{A}$	A set
$A \cap B$	the intersection of $A$ and $B$
$A'$	the complement of $A$
$A \rightarrow B$	$A$ is mapped on to $B$
$p \wedge q$	$p$ and $q$
$p \rightarrow q$	$p$ implies $q$
$\vec{OX}$	vector $OX$
$\bar{A}$	transposed matrix of $A$
$\sigma$	standard deviation
$\Rightarrow$	implies
$Z^+$	the set of positive integers
$\forall$	for any numbers
$a'$	the inverse of $a$
$I$	identity element
$\emptyset$	the null set
$\subset$	is contained in
$A \cup B$	the union of $A$ and $B$
$U$	the universal set (universe of discourse)
$\sim p$	negation of $p$
$p \vee q$	$p$ or $q$ or both
$p \leftrightarrow q$	$p$ implies $q$ and $q$ implies $p$
$A^{-1}$	inverse of $A$
$\mathbf{a}$	vector $\mathbf{a}$
$\supset$	has as one of its subsets
$\Leftrightarrow$	implies and is implied by
$\oplus$	operation.



## DEFINITIONS

Abelian group	A group which has a binary operation which is commutative.
algebra	A system is an algebra if it is provided with binary operations of addition and multiplication, and a scalar multiplication which make it both a vector space and a ring.
associative property	For (i) addition $(x + y) + z = x + (y + z)$ (ii) multiplication $(x \times y) \times z = x \times (y \times z).$
binary operation	An operation combining two elements.
binary scale	A number system containing only 0 and 1.
Boolean algebra	A class of statements and their logical relations. The algebra of sets.
closure property	The property that the result of combining any two elements is also a member of the set.
commutative property	For (i) addition $x + y = y + x.$ (ii) multiplication $x \times y = y \times x.$
conjunction	Both $a$ and $b.$
de Morgan's laws	(i) $(a + b)' = a'b'$ (ii) $(ab)' = a'b'.$
disjunction	Either $a$ or $b$ or both.
distributive property	Multiplication is distributive with respect to addition if $x \times (y + z) = x \times y + x \times z.$
domain of a function	The set of all the first members of the ordered pairs in the function.
function	A relation in which no two different pairs have the same first member.
field	A ring becomes a field if it has a unity element for multiplication and contains a reciprocal for every element except 0.

## group

### identity property

### inverse operation

### isomorphic group

### law of absorption mapping

### many-to-one correspondence

### matrix

### singular matrix

### transpose of a matrix

A system is a group if it has a binary operation that is associative, has an identity element for the operation, and has an inverse for every element.

That operation which leaves an element as before.  $I$  is an identity element for the operation  $\oplus$  if  $I \oplus X = X \oplus I = X$ , for every  $X$ .  $I$  is 0 for addition and 1 for multiplication.

$A$  is the inverse of  $B$  with respect to the operation  $\oplus$  whose identity element is  $I$ , if  $A \oplus B = B \oplus A = I$ . For (i) addition,  $a + (-a) = 0$ ,

where  $(-a)$  is the additive inverse of  $a$ . The inverse of an integer is its negative.

(ii) multiplication,

$$a \times \frac{1}{a} = 1, a \neq 0, \text{ where } \frac{1}{a}$$

is the multiplicative inverse of  $a$ . The inverse of an integer is its reciprocal.

Two groups are isomorphic if they have the same structure or form.

$$a + ab = a.$$

A function.

A single object is the image of more than one object.

An array of elements. An operator.

A matrix whose determinant is

zero. The matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is singular if  $ad - bc = 0$ .

If  $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ , the transposed matrix

$$\bar{A} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$



unit matrix	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .
zero matrix	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .
modulo arithmetic	A finite arithmetic. For example, the integers modulo 4 are 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, . . .
negative	$A$ is the negative of $B$ if $A + B = B + A = 0$ .
one-to-one correspondence	Each element of the range is matched with one and only one element of the domain.
ordered pair	A pair of elements arranged in a certain order.
range of a function	The set of all the second members of the ordered pairs in the function.
reciprocal	$A$ is the reciprocal of $B$ if $A \cdot B = B \cdot A = 1$ .
relation	A set of ordered pairs.
ring	A system is a ring if it has two associated binary operations called addition and multiplication, is an Abelian group with respect to addition, and if multiplication is distributive with respect to addition.
set	A collection of elements distinguishable from non members and from each other.
complement of set $A$	Those elements in the universal set which are not in set $A$ .
elements of a set	The members of a set.
intersection of two sets $A$ and $B$	Those elements which are in both set $A$ and set $B$ .
null set	The empty set, $\emptyset$ , or $\{ \}$ .
subset	$A$ is a subset of set $B$ if every element of $A$ is also an element of set $B$ .
union of two sets $A$ and $B$	Another set of elements which are in set $A$ or set $B$ or both.
universal set	The set of all the elements, called $U$ .

truth table  
 unity element  
 vector  
 vector space

A table showing whether or not a sentence is logically true.  
 $U$  is a unity element for multiplication if  $U \times X = X \times U = X$ , for every  $X$ .

A directed line segment.  
 A system is a vector space if it is an Abelian group with respect to addition, is subject to a scalar multiplication by elements from an associated field of scalars, and if the scalar multiplication obeys the laws:

$$\begin{aligned} r.(a + b) &= r.a + r.b \\ (r + s).a &= r.a + s.a \\ r.(s.a) &= (r.s).a \\ 1.a &= a. \end{aligned}$$

Venn diagram

A closed curve used to represent a set.

zero element

$A$  is a zero element for addition if  $A + X = X + A = X$ , for every  $X$ .



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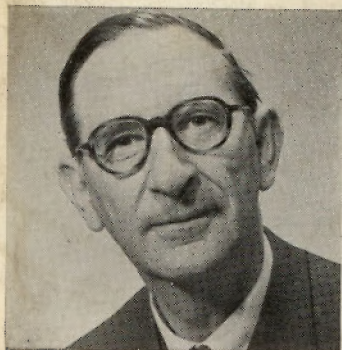
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C. C. T. Baker was born in Monmouthshire in November 1907. He was educated at Aberdare Grammar School, 1921-1926, and at The University College of Wales, Aberystwyth, 1926-1930, and was a scholarship holder at both places. He obtained the degree of B.Sc. in Mathematics and Physics in 1929, and the Diploma in Education in 1930.

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